# MODELLING
# OF OPTICAL IMAGE TRANSFORMATIONS
# BY MEANS OF 4x4 MATRICES

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## 1 Introduction

High-accuracy positioning of industrial manipulators is usually accomplished by means of laser-interferometric measurement systems. In such systems, beams of coherent light are subject to refractions and reflections in prisms and straightedges attached to various stage subassemblies, both moving and stationary. While analysing such a metrology system, one usually aims at determining the effect of both the stage motion and deviations from the layout geometry on the indications of the interferometers. Many ray tracing tools exist which perform this task numerically, calculating the beam position and orientation at its intersection with each reflection and refraction plane for a predefined configuration of prism and beam geometry. The problem with this approach is that it often requires a laborious layout definition, while the results it produces are purely numerical and thus do not offer a deeper insight into the effects involved.

Modelling techniques for rigid body transformations in 3D space are well known and utilise either 4x4 homogenous transformations or quaternions. This paper argues that certain 4x4 matrices can also represent image transformations associated with optical reflections and refractions on flat planes in 3D space. By representing prisms as matrices, the new method allows analysing their sequences without the need for ray tracing. It also presents an algebraic connection between those image transformation matrices and the motion applied to the reflecting and refracting planes, as well as prisms. The Authors are not aware of any references to 4x4 matrix formulation for purposes of optical analysis and claim that such formulation is innovative in this context.

The new technique is applicable to flat reflection/refraction surfaces. In addition, extracting the optical path in glass becomes possible as an algebraic operation. The new method is particularly useful when employed for symbolic computation in packages such as Mathematica and Maple. The proofs of the postulated image transformations require only elementary algebra.

## 2 Optical Image

In the context of this document, the optical image of a point under certain transformation will be the limit of the virtual origin of an infinitesimally narrow bundle of rays having undergone the transformation in question. The optical image of a vector, line or plane shall be constructed by aggregation of images of their member points.

### 2.1 Reflection

The optical image of a point under reflection is positioned on the other side of the reflection plane, at the same distance to it as the original, the segment connecting the image and the original being orthogonal to the reflection plane. It is postulated the a reflection plane satisfying the equation: $ax + by + cx + d = 0$ introduces an image transformation, which can be expressed as a 4x4 matrix:

$$\mathbf{K}_{refl} = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} -a^2+b^2+c^2 & -2ab & -2ac & -2ad \\ -2ab & a^2-b^2+c^2 & -2bc & -2bd \\ -2ac & -2bc & a^2+b^2-c^2 & -2cd \\ 0 & 0 & 0 & a^2+b^2+c^2 \end{pmatrix} \tag{1}$$

When multiplied by vector or point coordinates associated with the beam origin or direction, the 4x4 transformation yields their optical images in refraction or reflection. By the convention common in the traditional 4x4 spatial transformations, a unity is appended to point coordinates and a zero to vector coordinates to form a column vector of the required size.

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \mathbf{K}_{refl} \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} v'_x \\ v'_y \\ v'_z \\ 0 \end{pmatrix} = \mathbf{K}_{refl} \cdot \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix} \tag{2}$$

Please observe that the reflection transformation is its own inverse, what might be expected (reflecting twice leaves the original unchanged). Furthermore, the determinant of the reflection image transformation matrix is equal to −1, which represents the change of (left/right-) handedness:

$$\begin{gathered} \mathbf{K}_{refl}^{-1} = \mathbf{K}_{refl} \\ \det\left(\mathbf{K}_{refl}\right) = -1 \end{gathered} \tag{3}$$

To prove the correctness of the transformation, it is sufficient to show the proper distance equivalence and orthogonality for any plane $\mathbf{s} = (a, b, c, d)^T$ and point $\mathbf{p} = (x, y, z, 1)^T$, taking only 3 coordinates where a vector product necessitates it (we omit the complete derivation for brevity):

$$\begin{gathered} \mathbf{s}\cdot\mathbf{p}' + \mathbf{s}\cdot\mathbf{p} = \mathbf{s}\cdot\left(\mathbf{K}(\mathbf{s})\cdot\mathbf{p} + \mathbf{p}\right) = \mathbf{s}\cdot\left(\mathbf{K}(\mathbf{s}) + \mathbf{I}\right)\cdot\mathbf{p} = \mathbf{0}\cdot\mathbf{p} = 0 \\ \left(\mathbf{p}' - \mathbf{p}\right)^{[3]} \times \mathbf{s}^{[3]} = \left(\mathbf{K}(\mathbf{s})\cdot\mathbf{p} - \mathbf{p}\right)^{[3]} \times \mathbf{s}^{[3]} = \mathbf{0} \end{gathered} \tag{4}$$

### 2.2 Refraction

We put forward a hypothesis that in refraction, the optical image is located on the same side of the refraction plane, at a distance *n* times the original distance, *n* being the relative index of refraction. As before, the segment connecting the image and the original is orthogonal to the refraction plane. This is an approximation for small incidence angles, equivalent to assuming $\sin(x) \approx \tan(x)$ in the Snellius law of refraction. In general, the image location would depend on the incidence angle. The crucial observation is that up to the linear terms of the incidence angle, the image is angle-invariant, just like in a lens.

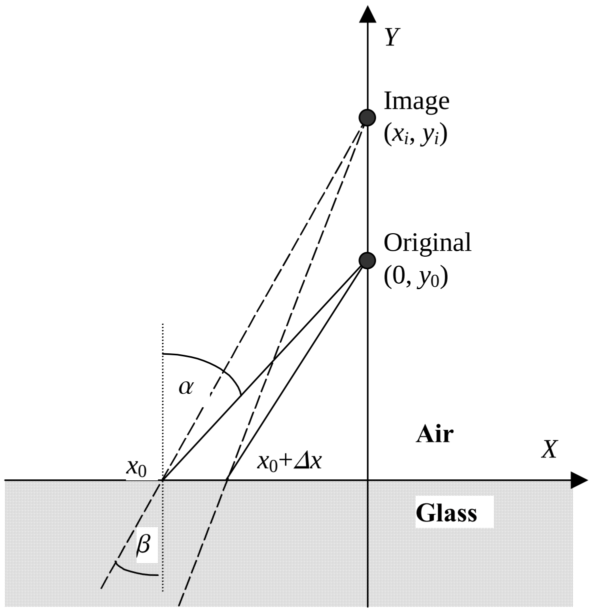


**Figure 1 The construction of an image in refraction.**

It can be shown (cf Figure 1) that for two rays originating from the same point $(0, y_0)$ and refracting at $(x_0, 0)$ and $(x_0+\Delta x, 0)$ the intersection of their extensions has a limit as their separation $\Delta x$ approaches 0, which is given by:

$$\begin{aligned} x_i &= -x_0\left(1-\frac{1}{n^2}\right)\tan^2\alpha \approx 0 \\ y_i &= ny_0\left[1+\left(1-\frac{1}{n^2}\right)\tan^2\alpha\right]^{\frac{3}{2}} \approx ny_0 \end{aligned} \tag{5}$$

By analogy to the reflection image transformation, we postulate that a refraction plane satisfying the equation: $ax + by + cx + d = 0$, introduces the following image transformation:

$$\mathbf{K}_{refr} = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} na^2+b^2+c^2 & (n-1)ab & (n-1)ac & (n-1)ad \\ (n-1)ab & a^2+nb^2+c^2 & (n-1)bc & (n-1)bd \\ (n-1)ac & (n-1)bc & a^2+b^2+nc^2 & (n-1)cd \\ 0 & 0 & 0 & a^2+b^2+c^2 \end{pmatrix} \tag{6}$$

$\mathbf{K}_{refr}$ can be applied in a manner analogical to $\mathbf{K}_{refl}$ in (2). Please also observe that $\mathbf{K}_{refl}$ is a special case of $\mathbf{K}_{refr}$, when $n = -1$. Furthermore, inverting the transformation matrix is equivalent to inverting the refraction index, as changing the medium from air to glass and back leaves the original unaltered, while the determinant predictably indicates a stretch in one direction:

$$\begin{aligned} \mathbf{K}_{refl}^{-1}(n) &= \mathbf{K}_{refl}\left(\frac{1}{n}\right) \\ \det\left(\mathbf{K}_{refr}\right) &= n \end{aligned} \tag{7}$$

As before, to prove the correctness of the transformation, it is sufficient to show the proper $n$:1 distance relation and orthogonality for any plane $\mathbf{s} = (a, b, c, d)^T$ and point $\mathbf{p} = (x, y, z, 1)^T$, taking only 3 coordinates where a vector product necessitates it (we omit the full derivation for brevity):

$$\begin{aligned} &\mathbf{s}\cdot\mathbf{p}' - n\cdot\mathbf{s}\cdot\mathbf{p} = \mathbf{s}\cdot\left(\mathbf{K}(\mathbf{s})\cdot\mathbf{p} - n\cdot\mathbf{p}\right) = \mathbf{s}\cdot\left(\mathbf{K}(\mathbf{s}) - n\cdot\mathbf{I}\right)\cdot\mathbf{p} = \mathbf{0}\cdot\mathbf{p} = 0 \\ &\left(\mathbf{p}'-\mathbf{p}\right)^{[3]}\times\mathbf{s}^{[3]} = \left(\mathbf{K}(\mathbf{s})\cdot\mathbf{p}-\mathbf{p}\right)^{[3]}\times\mathbf{s}^{[3]} = \mathbf{0} \end{aligned} \tag{8}$$

### 2.3 Transformation of a Plane

In order to obtain the image $\mathbf{s}' = (a', b', c', d')^T$ of a plane $\mathbf{s} = (a, b, c, d)^T$ under reflection or refraction **K**, the image transformation matrix has to be *inverted and transposed*:

$$\mathbf{s}' = \left(\mathbf{K}^{-1}\right)^T \mathbf{s} \tag{9}$$

To prove it, we first observe that if a point $\mathbf{p} = (x, y, z, 1)^T$ satisfies the equation of the original plane, the scalar product $\mathbf{s}^T \cdot \mathbf{p} = 0$. The image $\mathbf{p}' = (x', y', z', 1)^T$ of point **p** under **K** is given as $\mathbf{p}' = \mathbf{K} \cdot \mathbf{p}$ and it must satisfy the equation of the transformed plane $(\mathbf{s}')^T \cdot \mathbf{p}' = 0$:

$$\left(\mathbf{s}'\right)^T \mathbf{p}' = \left[\left(\mathbf{K}^{-1}\right)^T \mathbf{s}\right]^T \mathbf{K} \cdot \mathbf{p} = \mathbf{s}^T \mathbf{K}^{-1} \mathbf{K} \cdot \mathbf{p} = \mathbf{s}^T \mathbf{p} = 0 \tag{10}$$

### 2.4 Prism and Optical Path Transformation

A prism is a sequence of planes; it therefore possesses an image transformation being the product of image transformation matrices associated with the individual refraction and reflection planes. Due to the chosen left-associativity of the **K** operator, the order of matrices is the reverse temporal order in which the beam encounters the planes on its way:

$$\mathbf{K}_{prism} = \mathbf{K}_{refr}\left(\mathbf{s}_N, \frac{1}{n}\right) \cdot \mathbf{K}_{refl}\left(\mathbf{s}_{N-1}\right) \cdot \ldots \cdot \mathbf{K}_{refl}\left(\mathbf{s}_2\right) \cdot \mathbf{K}_{refr}\left(\mathbf{s}_1, n\right) \tag{11}$$

The first refraction plane (from air to glass) must be taken with the relative index of refraction *n*, the last refraction plane (from glass to air) with the relative index $1/n$. Naturally, the transformation matrices for all prisms participating in a given optical path can be multiplied by each other to produce a path transformation:

$$\mathbf{K}_{path} = \mathbf{K}_M \cdot \mathbf{K}_{M-1} \cdot \ldots \cdot \mathbf{K}_2 \cdot \mathbf{K}_1 \tag{12}$$

We can utilise the path transformation matrix to obtain the image of, for instance, an interferometric beam source or target (understood as point, vector or plane). This will allow analysing beam walking, angular deviations and other properties, both symbolically and numerically.

## 3 Motion Effect

### 3.1 Prism Motion

To determine the image transformation matrix of a prism to which a 4x4 spatial transformation **T** has been applied, one would normally apply **T** to all the reflection and refraction planes, construct a new chain of **K** matrices and multiply them out. (Applying a spatial transformation to a plane obeys the same rule as in (9), since the proof was not restricted to optical transformations.)

$$\mathbf{K} = \mathbf{K}\left(\left(\mathbf{T}^{-1}\right)^T \mathbf{s}_N\right) \cdot \mathbf{K}\left(\left(\mathbf{T}^{-1}\right)^T \mathbf{s}_{N-1}\right) \cdot \ldots \cdot \mathbf{K}\left(\left(\mathbf{T}^{-1}\right)^T \mathbf{s}_1\right) \tag{13}$$

However, this is a very laborious process and it fails quickly in symbolic computation. We will show that, for a spatially transformed prism, **K** is a direct function of **T**. It cannot be a function of the prism's individual orientation angles *Rx*, *Ry*, *Rz*, since those angles depend on the orientation convention (e.g. ZYX, XYZ, etc.), while the result cannot exhibit any such dependency. It is possible to prove that $f(\mathbf{T})$ must have the following algebraic form:

$$\mathbf{K} = f\left(\mathbf{T}\right) = \mathbf{T} \cdot \mathbf{K}_0 \cdot \mathbf{T}^{-1} \tag{14}$$

(The proof is based on the observation that applying an additional spatial transformation $\mathbf{\Delta T}$ to the prism and to the original must result in a similarly rotated image, as it is equivalent to transforming our frame of reference.) This result shows that every prism, irrespective of its number of surfaces, can be associated with a constant image transformation $\mathbf{K}_0$, which is derived at its nominal position and orientation. The significance of this fact is that $\mathbf{K}_0$ usually has fewer independent variables than the number of surfaces would indicate. Moreover, both the motion and the surface misalignment become apparent in a chain of matrix multiplications and are thus amenable to standard analytical methods, such as differentiation.

## 4 Time-Image

One should realise that the optical image transformation in refraction *contracts* the optical path in glass, instead of *expanding* it, which is what one would normally expect. For that reason, while perfectly suitable for analysing beam walking, it is rather inconvenient for deriving optical path lengths. In this section we present an alternative refraction transformation, one which straightens out the optical path while correctly accounting for the changes of the optical medium in an automatic and transparent manner.

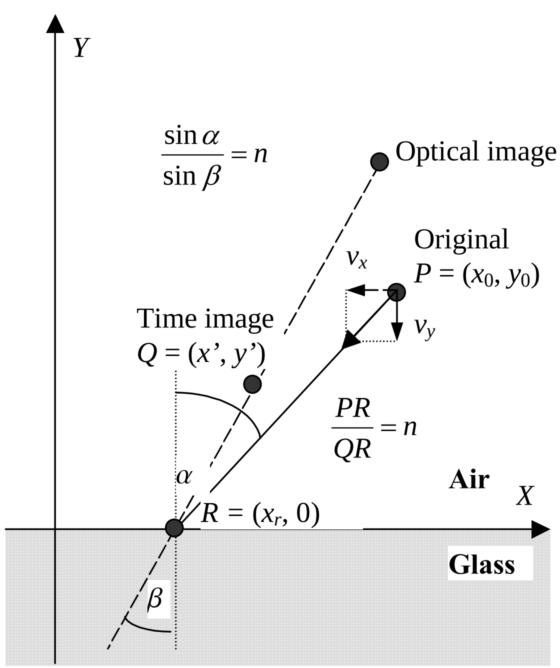


Figure 2 The construction of a time-image in refraction.

We introduce the *time-image* of a point under refraction as the virtual source of rays having undergone the refraction and satisfying the condition that **the time the ray travelled from the original to the refraction point is equal to the time it would have travelled from the image to the refraction point if it was immersed in the new medium**. In this manner, the optical path length from the original in two media is equal to the optical path length from the image in the new medium. Such a definition presupposes a given direction of the image-forming beam. Please observe that for an air-to-glass refraction, the traditional optical image is roughly $n$ times further away from the glass surface than the original, while the time-image is roughly $n$ times closer, where $n$ is the refraction index.

It can be shown (cf Figure 2) that the time-image of an original point $(x_0, y_0)$, projected by a beam arriving at an incidence angle $\alpha$ is given by the following equations:

$$x' = x_0 - y_0\left(1 - \frac{1}{n^2}\right)\tan\alpha$$
$$y' = \frac{y_0}{n}\sqrt{1 + \left(1 - \frac{1}{n^2}\right)\tan^2\alpha} \tag{15}$$

To determine the time-image transformation of an arbitrary original point **p** = ($x_0$, $y_0$, $z_0$) in a 3-dimensional space, for an arbitrary refraction plane given by the equation $ax + by + cx + d = 0$, and an arbitrary beam direction **v** = ($v_x$, $v_y$, $v_z$), we first have to transform spatially the original, the plane and the beam to the nominal arrangement (Figure 2), apply the time-image operation and transform the image back to chosen arrangement. For brevity, we present here only the final result:

$$\mathbf{K} = \mathbf{I} + \left[\frac{\left(\gamma - \frac{1}{n^2}\right)}{\left(a^2 + b^2 + c^2\right)}\begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix} - \frac{\left(1 - \frac{1}{n^2}\right)}{\left(av_x + bv_y + cv_z\right)}\begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix}\right] \cdot \begin{pmatrix} a & b & c & d \end{pmatrix} \tag{16}$$

where:

$$\gamma = \frac{1}{n}\sqrt{1 + \left(1 - \frac{1}{n^2}\right)\tan^2\alpha} \tag{17}$$

All the significant relations which we derived for the optical image, such as (2), (9) and (11), remain valid for the time-image as well. However, now the transformation matrix does depend on the incidence angle $\alpha$ and one has to transform the beam direction as well. The most practical approach is to operate on beam wavefront planes by setting $v_x = a$, $v_y = b$ and $v_z = c$ in (16), as the beam direction is perpendicular to the wavefront by definition.

# 5 Conclusions

Our paper shows that optical refraction and reflection on flat planes can be represented by means of a class of 4x4 matrices. When multiplied by vector or point coordinates associated with the beam origin, direction or wavefront plane, the 4x4 transformation yields their images in refraction or reflection. In this manner, the successive refractions and reflections the beam encounters on its way can be represented as a matrix product, while characteristic 4x4 matrices can be attributed to specific prism designs. Significantly, there exists a relationship between the optical 4x4 transformations and the homogenous 4x4 transformations, which are commonly used for modelling of rigid body motions in 3D space. This relationship allows for a seamless algebraic treatment of motion and optical phenomena in stage interferometry.

Furthermore, the time-image transformation we have introduced automatically accounts for the different media along the optical path length. This approach obviates a conscious consideration of the intersection between the beam and the refraction plane, thus offering “ray tracing without tracing rays”. Finally, the 4x4 optical transformations allow a degree of symbolic analysis which could not be afforded with traditional ray tracing. This, in turn, leads to explicit model expressions for the indicated optical path lengths which are functions of motion and geometry errors. Such expressions can be useful as coordinate transformations required for motion control and as parametric models in calibration of the position metrology.