Radius of curvature uncertainty: Nonlinear measurand and treatment

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Radius of curvature measurement

Many methods are available to determine the curvature of a spherical optical surface:

- Templates/test plates
- Spherometers
- **Optical bench with Fizeau (or T-G) interferometer**

Importance:

- Key variable in optical system design that affects imaging performance.
- Radii error often compensated by respacing lens assembly to obtain desired wavefront

Industrial disagreement for test plate round robin suggests need for better understanding of measurement model and uncertainty.
Optical bench radius measurement

Reference surface

Interferometer (Fizeau setup)

Cat’s eye

Confocal

Transmission sphere

Zernike power \((r^2)\) term from phase maps.

Wavefront focus at test optic surface

Wavefront curvature matches test optic

DMI

Radius

\(a_2^0\)
Optical bench radius measurement

Traceable radius measurement requires (ISO GUM, NIST 1297):

- Bias corrections
- Combination of uncertainty sources to yield $u_c$, which represents $1\sigma$ of measurand
- Careful definition of the measurand
  - HTM formalism
  - Vector equation used to define radius
  - HTM used to determine test artifact location after translation from confocal to cat’s eye

Analysis:

- Measurand is nonlinear

- Cannot simply insert expectation values for uncertainty contributors to determine best estimate of the measurand.

- Monte Carlo simulation applied
Optical bench radius measurement

Many uncertainty and bias sources exist:
1. Slide motion uncertainties (confocal to cat’s eye)
2. Displacement gage (Abbe, cosine, transducer-specific such as environment for DMI)
3. Optical (wavefront aberrations, artifact surface figure, null locations, aperture variation)

Previous work focused on RSS combination of uncertainty sources using law of propagation of uncertainty, based on a first-order Taylor series expansion of $Y = f(X_1, X_2, \ldots X_N)$.

$$ u_c^2(y) = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j) $$

**Objective:** develop equation that represents radius as function of error sources and analyze uncertainty.
Measurand definition

Coordinate frame identification for the confocal position. Reference (r) and stage (s) frames overlap at confocal location.

Probe location (geometric focus) in reference frame

Artifact center location in stage frame

From confocal, move to cat’s eye location.
Measurand definition

Two scenarios for movement to cat’s eye shown.

Perfect translation to cat’s eye along $z_r$ axis. Gage reading at cat’s eye, $d_{ce}$, is equal $R$

Straightness error, $\delta_x$, causes gage reading to be $< R$ (bias introduced if using $d_{ce}$)

Rather than gage reading, use vector equation to define $R$. 
Measurand definition

Again see imperfect motion from confocal to cat’s eye. Now define $R$ as measurand, rather than using $d_{ce}$.

$$R = |\vec{R}| = \sqrt{|r\vec{X}_p - r\vec{X}_{ce}|^2}$$

Must determine vectors: $r\vec{X}_p$ $r\vec{X}_{ce}$
Measurand definition

\[ \vec{R} = \vec{r} \hat{x}_p - \vec{r} \hat{x}_{ce} \]

\( \vec{r} \hat{x}_p \) 3-element vector describing probe (PMI focus) in reference frame.

\( \vec{r} \hat{x}_{ce} \) Describe location of artifact center (at cat’s eye) in reference frame using HTM.

\[ r \hat{x}_{ce} = T_s r \hat{x}_{ce} \]

Rotation matrix

\[ r T_s = \begin{bmatrix} 1 & -\epsilon_y & \epsilon_z & \delta_x \\ \epsilon_y & 1 & -\epsilon_x & \delta_y \\ -\epsilon_z & \epsilon_x & 1 & \delta_z + d \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

Small, rigid body motions assumed.
Populate HTM using measured roll, pitch, yaw, and straightness error motions for stage (depend on stage location in z).

Artifact center in stage frame defined using probe offsets in reference frame and ability to identify confocal (recall stage and reference frames initially coincident).

\[
\vec{R} = r\vec{x}_p - r\vec{x}^{ce}_A
\]

\[
\vec{s}_A \quad R \quad \left[ \begin{array}{c} dx^{cf} \\ dy^{cf} \\ dz^{cf} \\ 1 \end{array} \right]
\]

\[
= \left[ \begin{array}{c} -dx^{cf} \\ -dy^{cf} \\ -dz^{cf} \\ 1 \end{array} \right] + \left[ \begin{array}{c} \delta_x \\ \delta_y \\ \delta_z \\ d \end{array} \right]
\]

I: Null confocal
II: Abbe
III: Straightness/cosine (linear term)
IV: Gage reading
Measurand definition

Expand to find $R$, let probe offset in $z$ and positioning error be zero.

\[ r_z = 0 \quad \text{and} \quad \delta_z = 0 \]

\[
R = \left( d_{ce}^2 + \delta_x^2 + \delta_y^2 + dx_{cf}^2 + dy_{cf}^2 + dz_{cf}^2 + 2dx_{cf} \delta_x + 2dy_{cf} \delta_y + 2dz_{cf} d_{ce} + \varepsilon_z^2 \left( r y_p^2 + 2 r y_p dy_{cf} + dy_{cf}^2 \right) + \varepsilon_z^2 \left( r x_p^2 + 2 r x_p dx_{cf} + dx_{cf}^2 \right) + \varepsilon_y^2 \left( r x_p^2 + 2 r x_p dx_{cf} + dx_{cf}^2 \right) + \varepsilon_x^2 \left( r y_p^2 + 2 r y_p dy_{cf} + dy_{cf}^2 \right) + \varepsilon_x^2 \left( dz_{cf}^2 \right) + \varepsilon_y^2 \left( dz_{cf}^2 \right) + 2 \varepsilon_z \delta_y \left( y_p + 2 \varepsilon_x \delta_x \left( y_p \right) + 2 \varepsilon_x \left( y_p d_{ce} - 2 \varepsilon_y x_p d_{ce} + 2 dy_{cf} \varepsilon_x d_{ce} - 2 dx_{cf} \varepsilon_y d_{ce} + 2 dy_{cf} \varepsilon_z \left( x_p - \delta_x \right) - 2 dx_{cf} \varepsilon_z \left( y_p - \delta_y \right) \right) + 2 dz_{cf} \varepsilon_x \left( y_p - \delta_y \right) - 2 dz_{cf} \varepsilon_y \left( x_p - \delta_x \right) - 2 \varepsilon_x \varepsilon_y \left( x_p \left[ y_p + x_p dy_{cf} + r x_p dx_{cf} + dx_{cf} dy_{cf} \right] \right) - 2 \varepsilon_x \varepsilon_z \left( x_p dz_{cf} + dx_{cf} dz_{cf} \right) - 2 \varepsilon_y \varepsilon_z \left( r y_p dz_{cf} + dy_{cf} dz_{cf} \right) \right) \left( Yikes! \right)
\]
Analytical expectation value

Assume errors are zero on average with no cosine error (no linear term in straightness) to give simplified expression.

\[
R \approx \left( d_{ce}^2 + \delta_x^2 + \delta_y^2 + dx_{cf}^2 + dy_{cf}^2 + dz_{cf}^2 + \epsilon_z^2 \left( r y_p^2 + dy_{cf}^2 \right) + \epsilon_z^2 \left( r x_p^2 + dx_{cf}^2 \right) + \epsilon_y^2 \left( r x_p^2 + dx_{cf}^2 \right) + \epsilon_x^2 \left( d_{cf}^2 \right) + \epsilon_y^2 \left( d_{cf}^2 \right) \right)^{1/2}
\]

To find expectation value for \( R \), may be tempted to substitute expectation value for each input on right hand side of equation.

\[
\langle R \rangle = \sqrt{\sum \langle a^2 \rangle}
\]

Not valid due to square root (nonlinear operator).

Consider parameter \( x \) with mean zero noise \( \Delta_x \).

\[
\langle x \rangle = \sqrt{\langle (x + \Delta_x)^2 \rangle} \neq \sqrt{\langle (x + \Delta_x)^2 \rangle} = \sqrt{\langle x^2 \rangle + 2x\Delta_x + \Delta_x^2} = \bar{x}
\]

Nonzero
Analytical expectation value

Reality check: Consider that we have directly substituted the expectation value for each term on the right hand side to find \( \langle R \rangle \):

\[
R \approx \left( \frac{d_{ce}^2 + \delta_x^2 + \delta_y^2 + dx_{cf}^2 + dy_{cf}^2 + dz_{cf}^2}{1} \right)^{1/2}
\]

\[
= \left( \varepsilon_z^2 \left( r y_p^2 + dy_{cf}^2 \right) + \varepsilon_z^2 \left( r x_p^2 + dx_{cf}^2 \right) +
\varepsilon_y^2 \left( r x_p^2 + dx_{cf}^2 \right) + \varepsilon_x^2 \left( dz_{cf}^2 \right) + \varepsilon_x^2 \left( dz_{cf}^2 \right) \right)^{1/2}
\]

We’ve seen that \( \delta_x \) introduces a bias into \( d_{ce} \). The nonzero variance could serve as a bias correction… this is good.

Consider \( dz_{cf} \), ability to null confocal in \( z \). All other errors zero. For mean zero with \( \sigma_{dz} \), you’d expect \( R = d_{ce} \) and \( \sigma_R = \sigma_{dz} \).

However, for direct substitution of expectations values, a bias of \( \sigma_{dz}^2 \) would be generated… this is bad.
Monte Carlo expectation value

Monte Carlo simulation

• History is gaming tables at casinos in Monte Carlo (Hazelrigg)

• Use computer to generate random numbers (of stated distribution) that perturb the mean values of the input quantities ($d_{ce}$, error motions, offsets, confocal alignments)

• Calculate the output quantity, $R$ (used full equation)

• Repeat (100k points in our analysis)

Caveats:

Random number generation can be tricky (seed, algorithm).

Used Matlab™ $randn$ (normal) function; $2^{1492}$ values before repeating.
Monte Carlo expectation value

\[
\hat{R} = r \hat{x}_p - r T_s \left( r \hat{x}_p + \begin{bmatrix} dx^{cf} \\ dy^{cf} \\ dz^{cf} \\ 1 \end{bmatrix} \right) = \begin{bmatrix}
-dx^{cf} & -\varepsilon_z (r y_p + dy^{cf}) - \varepsilon_y (r z_p + dz^{cf}) & -\delta_x & 0 \\
-dy^{cf} & -\varepsilon_z (r x_p + dx^{cf}) + \varepsilon_x (r z_p + dz^{cf}) & -\delta_y & 0 \\
-dz^{cf} & \varepsilon_y (r x_p + dx^{cf}) - \varepsilon_x (r y_p + dy^{cf}) & -\delta_z & d \\
\end{bmatrix}
\]

I: Null confocal
II: Abbe
III: Straightness/cosine (linear term)
IV: Gage reading

Category I and III terms give SSD (biased) for \(x\) and \(y\) terms, but normal distributions for \(z\) terms (e.g., \(dz^{cf}\)).
Results for mean zero $\delta_x$ error, $\sigma_{\delta_x} = 25 \, \mu m$. SSD distribution is observed (12 nm bias).

Mean zero $dz^{cf}$ error, $\sigma_{dz} = 25 \, \mu m$. No bias observed.
Uncertainty evaluation

Two options include:

1. Analytical – Taylor series expansion of $R = f(X_1, X_2, \ldots X_N)$.

$$u_c^2(R) = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$$

Cumbersome expressions for complicated $R$ equation.

2. Monte Carlo simulation

Using this approach to find expectation value; can take standard deviation to be best estimate of $u_c(R)$. 
## Uncertainty evaluation

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<th>Input</th>
<th>Mean</th>
<th>$\sigma$</th>
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</thead>
<tbody>
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<td>50 $\mu$m</td>
</tr>
<tr>
<td>$r_y$</td>
<td>2 mm</td>
<td>50 $\mu$m</td>
</tr>
<tr>
<td>$r_z$</td>
<td>0</td>
<td>50 $\mu$m</td>
</tr>
<tr>
<td>$d_{ce}$</td>
<td>50/5 mm</td>
<td>50 nm</td>
</tr>
<tr>
<td>$dx^{cf}$</td>
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<tr>
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</tr>
<tr>
<td>$\varepsilon_z$</td>
<td>0</td>
<td>5 $\mu$rad</td>
</tr>
</tbody>
</table>

Monte Carlo analysis:

Input values shown (normal distributions assumed)

Two $d_{ce}$ values applied: 50 mm and 5 mm

Mean $R$ and $\sigma_R$ recorded.

Standard deviations held constant since they represent ability to measure.
Uncertainty evaluation

For 50 mm:
- \( \mu_c(R) \sim 5.0 \, \mu m \)
- \( \frac{\mu_c(R)}{R} \cdot 100\% \approx 0.01\% \)

For 5 mm:
- \( \mu_c(R) \sim 5.0 \, \mu m \)
- \( \frac{\mu_c(R)}{R} \cdot 100\% \approx 0.1\% \)
Case study: Abbe offset

(Top) zero Abbe offset setup; (bottom) nonzero x-direction Abbe offset.
Case study: Abbe offset

Gage reading is not equal to radius for nonzero x-direction Abbe offset coupled with small rigid body rotation about the y-axis.

If $\varepsilon_y = -25$ arc-sec, $R = 25$ mm; $d_{ce} = -25.012120$ mm for 100 mm offset. Similarly, if $\varepsilon_y = +25$ arc-sec, then $d_{ce} = -24.987880$ mm.

- Biased result if $R = 0 - d_{ce}$.
- Correct $R$ value obtained once $d_{ce}$, Abbe offset, and $\varepsilon_y$ are substituted in the HTM $R$ equation.
Case study: Abbe offset

Even if Abbe error compensated (e.g., by angular measurements), measurement uncertainty is still affected.

\[ R = 25 \text{ mm Monte Carlo simulation:} \]

1. **Zero Abbe offset**
   Mean zero \( \varepsilon_y \)
   \[ \sigma_{\varepsilon_y} = 25 \text{ arc-sec} \]
   \( d_{ce} = -25 \text{ mm} \)
   \( \sigma = 0 \text{ nm} \)

2. **100 mm offset**
   \( d_{ce} = -25 \text{ mm} \)
   \( \sigma = 12100 \text{ nm} \)

For \( \sigma < 1 \text{ nm} \), \( \sigma_{\varepsilon_y} < 0.002 \text{ arc-sec} \)…
Summary

• Vector equation for measurand \((R)\) in optical bench measurements defined.

• Shown that care must be taken in finding expectation value analytically.

• Monte Carlo simulations completed. Verified that some terms introduce SSD; vector equation corrects for biases.

• Mean and standard deviations calculated for sample cases.

• Abbe offset case study provided.

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