A SYSTEMATIC APPROACH TO THE MODELLING OF MEASUREMENTS FOR UNCERTAINTY ANALYSIS

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1 INTRODUCTION

The modern evaluation of measurement uncertainty is based on both, the knowledge about the measuring process and the (input) quantities that may influence the measurement result. The knowledge about these input quantities is to be expressed by means of appropriate probability distributions functions (pdf) for these quantities, whereas the knowledge about the measuring process is to be condensed to the so-called model equation. This equation mathematically interrelates the measurement *Y* and all relevant input quantities $X_1,...,X_N$:

$$Y = f_Y\left(X_1,...,X_N\right). \tag{1.1}$$

Since neither the *Guide to the Expression of Uncertainty in Measurement (GUM)* [1] nor other relevant uncertainty documents provide sufficient guidance on systematic modelling procedures to practitioners, modelling appears to be the most difficult problem in uncertainty evaluation. First approaches to a systematic and *GUM*-consisting modelling procedure were made by *Bachmair* [2], *Kessel* [3], *Kind* [4] and *Sommer* et al. [5; 6]. This paper reports the progress made and extends the previous work.

2 BASIC RELATIONSHIPS

The *GUM* concept for evaluating the uncertainty is based on the knowledge about the measuring process and the quantities which may have influence on the measurement result and their associated uncertainties. In accordance with the *GUM* concept [1] the (unavoidably incomplete) knowledge about each contributing input quantity is to be expressed by means of pdfs $g_{Xi}(\xi_i)$. The expectation value of the pdf is the best estimate of the value of the quantity,

$$x_i = \mathrm{E}\left[X_i\right] = \int_{-\infty}^{+\infty} g_{Xi}(\xi_i)\xi_i \, d\xi_i \, , \tag{2.1}$$

and its standard deviation is the uncertainty u_{xi} associated with this estimate,

$$u_{xi} = \left\{\mathrm{E}\left[\left(X_i - x_i\right)^2\right]\right\}^{1/2} = \left\{\int_{-\infty}^{+\infty} g_{Xi}\left(\xi_i\right)\left(\xi_i - x_i\right)^2 d\xi_i\right\}^{1/2} \tag{2.2}$$

Lower case Greek letters are used for the possible values of a quantity, $\xi \;\; for \;\; X$ and $\eta \;\; for \;\; Y$.

The pdfs for the input quantities are obtained from given information by utilizing the principle of maximum information entropy (pme) [7]. New information, e. g. from additionally measured data is included by using the *Bayes* theorem [8]. The standard-*GUM* procedure condenses the pme and parts of the *Bayesian* concept to the evaluation methods of *type-A* (statistical analysis of information from repeated measurements) and *type-B* (evaluation by any other means). In both cases, one obtains unambiguously a pdf for each input quantity.

The pdf for the measurand Y, $g_Y\left(\eta\right)$, is given by the integral

$$g_Y(\eta) = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} g_{X1,...,XN}(\xi_1,...\xi_N)\delta\left(\eta - f_Y(\xi_1,...,\xi_N\right)d\xi_1...d\xi_N \tag{2.3}$$

where f_Y is the functional relationship of the values of the involved quantities, ξ, with the respective value of the measurand, η. From the above pdf $g_Y(\eta)$, the expectation value of the measurand $y = \mathrm{E}[Y]$ and its associated uncertainty u_y can be calculated as follows:

$$y = \int_{-\infty}^{+\infty} g_Y(\eta)\eta d\eta, \text{ and } u_y = \int_{-\infty}^{+\infty} g_Y\left(\eta\right)\left(\eta - y\right)^2 d\eta. \qquad (2.4)\ (2.5)$$

This way of uncertainty determination by means of pdf propagation is illustrated in Fig. 1. But because equation (2.3) can analytically be computed in fairly simple cases only, modern uncertainty evaluation uses the *Monte-Carlo Method* as an integration technique [9]. *Monte-Carlo* techniques are based on the sampling of the cumulative input pdfs: With uniform probability, probability values $G_{Xi}\left(\xi_i\right)$ of the input quantities are selected. Then, the related arguments ξ_i of each quantity are to be combined in accordance with the model equation for the values, $y = f_Y\left(\xi_1,...,\xi_N\right)$, where f_Y is identical with f_Y in equation (1.1) yielding values η_k. From a sufficient number of samples, a frequency distribution is obtained that approximates the probability distribution $g_Y\left(\eta\right)$ Fig. 2 illustrates the *Monte-Carlo* integration technique that handles linear and non-linear models alike.

In contrast to *Monte-Carlo* techniques, the standard-*GUM* procedure (see Fig. 3) [1] applies to linear or linearized models only. In practice, most dependencies on input quantities can be approximated by first-order *Taylor* series expansion within the range extended by the uncertainties associated with the values of the input quantities:

$$\eta = f\left(\xi\right) \cong f\left(x\right) + \sum_{i=1}^{N} c_i\left(\xi_i - x_1\right) \qquad (2.6)$$

where η denotes a possible value of the output quantity Y, and ξ is used as abbreviation for $\xi_1,...,\xi_N$, and x for $x_1,...,x_N$. The c_i are called sensitivity coefficients and given by

$$c_i = \frac{\partial f(X_1,...,X_N)}{\partial X_i}\bigg|\forall\, \xi_i = x_i\,. \qquad (2.7)$$

Therefore, for a linear (or linearized) model equations, the expectation value of the output quantity Y is

$$Y = \mathrm{E}\left[X_i\right] = f_Y\left(x_1,...,x_N\right). \qquad (2.8)$$

The uncertainty associated with this expectation is obtained from the law of *Gaussian* uncertainty propagation:

$$u_y = \left\{\sum_{i=1}^{N} c_i^2 u_{xi}^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} c_i c_j u_{xixj}\right\}^{1/2}, \qquad (2.9)$$

where $u_{xixj} = u_{xi}u_{xj}\cdot r_{ij}$ is the covariance of the quantities X_i and X_j, and r_{ij} the correlation coefficient.

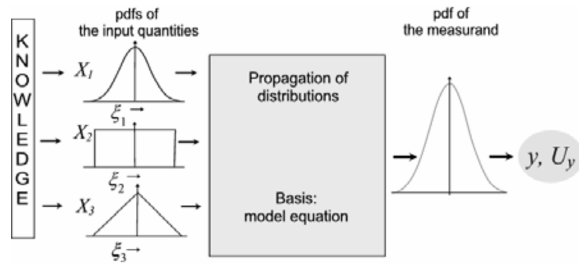

Fig. 1: Illustration of the concept of *pdf* progagation in uncertainty evaluation [5]. Symbols: Y – Measurand; $X_1,...,X_N$ – Input quantities; $y = \mathrm{E}\left[Y\right]$ – expectation value of the measurand; u_y – uncertainty associated with y

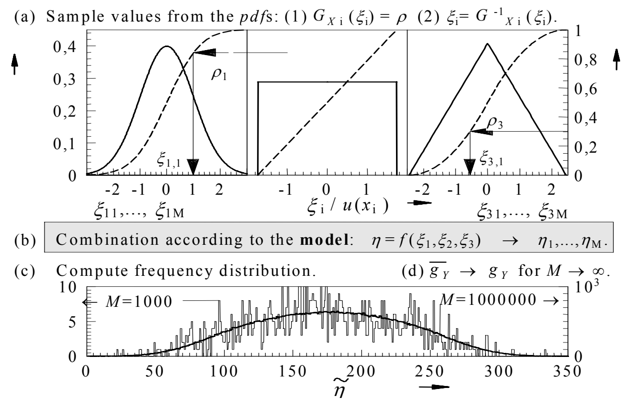

Fig. 2: Illustration of Monte-Carlo integration (a) The upper three graphs show (solid lines) the pdfs to the (input) quantities and (dashed lines) the respective cumulative distributions. (b) Combination of the arguments (c) The curves in the bottom part show computed frequency distributions for two sample sizes M. (d) Approximation of the pdf to the measurand

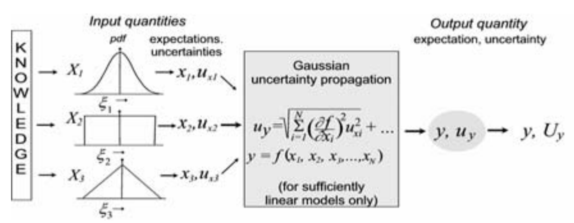

Fig. 3 Illustration of the concept of the standard-*GUM* procedure [1; 5]. Symbols see Fig. 1

From the above basic relationships on uncertainty evaluation, it may be concluded that – independent on the method utilized – the model equation is an indispensable prerequisite of modern uncertainty evaluation.

3 MODELLING PROCEDURE

3.1 Concept

The modelling concept presented [5; 6] is based on the idea of the *measuring chain* which constitutes the path of the measurement signal from cause to effect. In metrological practice, the following assumptions can be made:

- At least in narrow ranges around the operating points, the great majority of measuring systems and devices (elements) may be regarded to have linear characteristics and can be approximated by first-order Taylor series expansion respectively (see equation (2.6.)).
- The (steady-state) transmission behaviour of a measuring system is always related to well-adjusted and known operating conditions.
- The „real world of measurement" may be taken into consideration by means of deviations of the real influence quantities and other parameters.

On the above assumptions, in steady state, almost all functional elements or operational steps of a measuring system or process may be described by an approximately constant transmission factor and by *deviations* representing the imperfection of the measurement. Deviations may have impact on transmission factors and they may result in offsets of the outputs. Fig. 4 illustrates this concept of a perturbed transmission element that mathematically can be expressed by the following relationship:

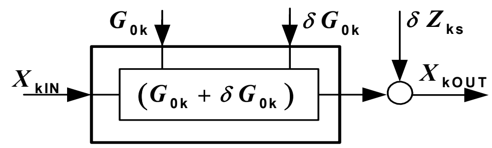

Fig. 4: Concept of a perturbed transmission element [5]

$$X_{KOUT} = X_{kIN}\left(G_{OK} + \delta G_{OK}\right) + \delta Z_K \tag{3.1}$$

where: X_{KIN} – quantity acting at the input of the element k ; X_{KOUT} – quantity at the output of the element k; G_{OK} – transmission factor; δ_{GOK} – parameter deviation δZ_k – parameter deviation.

The above concept allows to graphically describe the cause-and-effect relationship of a measurement at any operating point.

3.2 Standard modelling components

For the required graphical depiction of the cause-and-effect relationships of the measurements to be modelled (see 3.3), only three types of standard modelling components are employed:

- **Parameter sources** to provide or reproduce a measurable quantity.
- **Transforming units** that represent any kind of parameter processing and influencing.
- **Indicating units** to indicate their input quantities.

3.3 Modelling procedure

Based on the above presumptions, a straightforward and highly versatile modelling procedure has been developed [5; 6] that consists of the following elementary steps:

1st step: Description of the measurement, analysis of the measuring process by decomposing it in terms of standard modelling components (see 3.2) with a view to form the measuring chain.

2nd step: Graphical depiction of the cause-and-effect relationship for a fictitious ideal (unperturbed) measurement in terms of standard modelling components.

3rd step: Inclusion of all imperfections and effects that may perturb the fictions ideal measurement, such as, for example, influences, incomplete knowledge about parameters etc., and representation of the resulting cause-and-effect relationship graphically and, in turn, mathematically for the real measurement.

4th step: Identification and inclusion of possible correlation in the measuring chain [10]:
1st way: If correlation is caused by conjoint functional dependencies on a third quantity, such as, for example, on temperature, these dependencies are to be accounted. The way to do this is to introduce these

dependencies in the graphical and mathematical cause-and-effect relationship and, therewith, to dissolve correlation. If possible to go, this first way is to be preferred.
2nd way: Correlation is taken into account in accordance with the law of *Gaussian* uncertainty, propagation (see equation (2.9)). This way, however, requires the knowledge of the (estimated or experimentally determined) value of the correlation coefficient [10].

5th step: Inversion of the cause-and-effect relationship to derive explicitly the mathematical relationship between the output quantity and the relevant input quantities.

4 EXAMPLE

4.1 Modelling procedure

The modelling procedure is explained with the (simplified) example of the determination of a weighing value by direct weighing.

1st step:

- *Description of the measurement:* The weighing value of a weight of about 10 kg is to be determined by direct weighing taking one reading. The scale used is a 12 kg- range instrument (scale value: 0.1 g; maximum permissible error on verification, mpev = 0.5 g).
- *Measurand:* Weighing value W_X of the weight piece.
- *Measurement method:* Direct measurement.
- *Analysis of the measuring process:* The weight piece may be regarded as the parameter source, and the imperfect „coupling“ of the unknown weighing value with the scale (causes: air buoyancy, magnetic, susceptibility etc.) may be described by a transforming unit. The scale itself may be represented by an indicating unit.

2nd step: *Cause-and-effect relationship of the fictitious ideal measurement:*
Fig. 6 shows the graphical cause-and-effect relationship of the idealized (unperturbed) measurement.

3rd step: *Cause-and-effect relationship of the real measurement:*
Fig. 7 shows the graphical cause-and-effect relationship. The following imperfections have been introduced:

δ_{WCPL} – deviation due to eccentric loading, vibration, air convection, magnetic susceptibility etc.;

k_B – $k_B = \left(1-\rho_a\rho_x^{-1}\right)\left(1-\rho_{1.2}\rho_{8000}^{-1}\right)^{-1}$, air buoyancy factor, where ρ_a is the air density ρ_X is the density of the weight to be calibrated, and $\rho_{1.2} = 1.2kgm^{-3}, \rho_{8000} = 8000kgm^{-3}$;

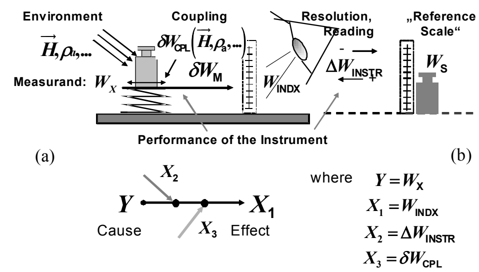

Fig. 5: (a) Simplified example: Direct determination of an unknown weight; (b) Visualisation of the cause-and-effect relationship by means of a graph. Symbols: W_X - measurand; δW_{CPL} - unknown deviation due to the imperfect „coupling“ of the measurand with the scale (causes: air buoyancy etc.); W_{INDX} – indicated quantity; ΔW_{INSTR} - instrumental error of the weighing instrument; W_S - standard weight; $\vec{H}$ - magnetic field strength; ρ_a - air density

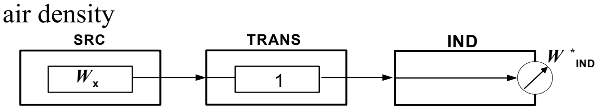

Fig. 6: Graphical depiction of the cause-and-effect relationship of the fictitious ideal weighing procedure in accordance with 4.1. Symbols: W_X - measurand; W^*_{IND} - indicated quantity

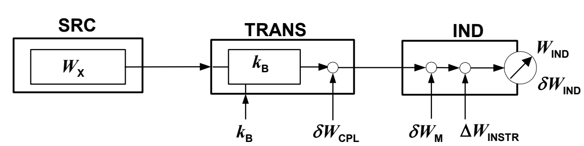

Fig. 7: Cause-and-effect relationship of the real measurement in accordance with the described example (clause 4.1) Symbols: see text

$\delta W_M(t_a)$ – deviation due to the susceptibility of the weighing instrument to ambient temperature t_a;

ΔW_{INSTR} – instrumental error of the weighing instrument;

δW_{IND} – deviation due to the limited resolution of the instrument.

In mathematical terms, the cause-and-effect relationship of the real measurement reads:

$$W_{IND} = W_X k_B + \delta W_{CPL} + \delta W_M + \Delta W_{INSTR} + \delta W_{IND} \qquad (4.1)$$

4th step: *Correlation:*
For the sake of simplification, all in involved quantities, parameters and observations are assumed to be independent of each other.

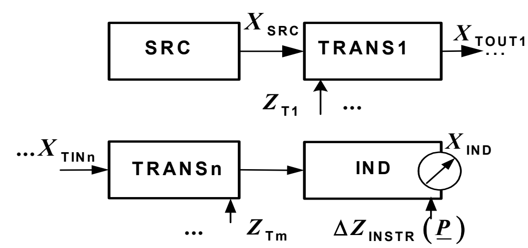

Fig. 8: Generic structure of the cause-and-effect relationship of direct measurements. Symbols: X_{SRC} – output quantity of the parameter source; $\underline{P}$ – measurement conditions; $Z_{T1},...,Z_{Tm}$ – additional input quantities, such as, for example transmission factors and deviations; X_{IND} – indicated quantity, – $\Delta Z_{INSTR}(\underline{P})$ instrumental error;

5th step: *Model equation*:
From the cause-and-effect relationship of the real measurement (see equation (4.1)), the following model equation is obtained:

$$W_X = \left(W_{IND} - \delta W_{IND} - \Delta W_{INSTR} - \delta W_M - \delta W_{CPL}\right) k_B^{-1} \qquad (4.2)$$

4.2 Evaluating the measurement uncertainty

Due to the almost linear model equation of the chosen example (see equation (4.2)), the standard-*GUM* Method may be used to determine the measurement uncertainty. After modelling the measurement, the most important step is to evaluate the involved input quantities W_{IND}, δW_{IND}, ΔW_{INSTR}, δW_M, δW_{CPL}, k_B. To each of these quantities, an appropriate pdf is to be assigned. W_{IND} will be clearly indicated. The values of δW_{IND} may be derived from the instrument's resolution and for ΔW_{INSTR} from the maximum permissible error stated for the instrument. The knowledge about δW_{CPL} *and* δW_M may be taken up from the manufacturers manual or from the requirements set up in the European standard *EN 45501*. k_B is to be estimated based on the knowledge about the ambient conditions and about the weight piece to be measured.

5 ROLE OF THE MEASUREMENT METHOD

The structure and the chaining sequence of the cause-and-effect relationship are determined by the *method of measurement* used. Direct measurements result in an unbranched chain of the components utilized (see Fig. 8).

Other methods are used to achieve higher accuracies and to ensure proper traceability of calibration results. These methods mostly result in branched cause-and-effect relationships. Examples are given with the direct comparison of two indicating measuring instruments and substitution method. Fig. 9 and 10 show the generic structures of their cause-and-effect relationship. When deriving the mathematical cause-and-effect relationship from block diagrams having branched structures, such as, for example, the above methods, for each branch a separate (partial) equation is to be derived [5]. Thereby, influences and imperfections of the commonly used parts of the paths should be assumed to be correlated.

6 CONCLUSION

Although if seems not possible to develop a theory that allows for a stringent construction of a model, it is, nevertheless, possible to achieve systematic modelling based on the presented concept. Systematic modelling may be seen as an important improvement of uncertainty evaluation . The particular concept presented is applicable to most areas of uncertainty evaluation of measurements performed. It is clearly structured in five elementary steps, and only three types of standard modelling components are employed.

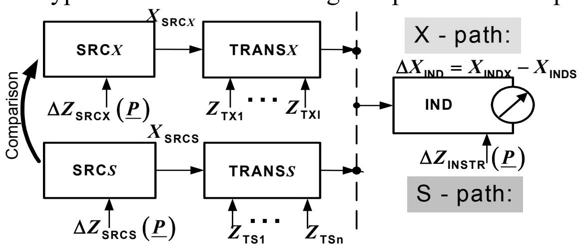

Fig. 9: Generic structure of the cause-and-effect relationship of a calibration of a material measures employing *substitution method*. Symbols: SRCX – material measure to be calibrated; SRCS – standard material measure; X_{SRCX} – output quantity of SRCX; $\Delta Z_{SRCX}(\underline{P})$ – measurand; $\underline{P}$ – calibration conditions; X_{SRCS} – output quantity of the standard; $\Delta Z_{SRCS}(\underline{P})$ – instrumental error of the standard used; TRANSX / TRANSS – transforming units; IND – indicating instrument (comparator); $\Delta Z_{INSTR}(\underline{P})$ – instrumental error of the comparator; $Z_{TX1}, \ldots, Z_{TXl}$ and $Z_{TS1}, \ldots, Z_{TSn}$ – additional input quantities; $\Delta X_{IND} = X_{INDX} - X_{INDS}$ – indication difference between the material measure and the standard

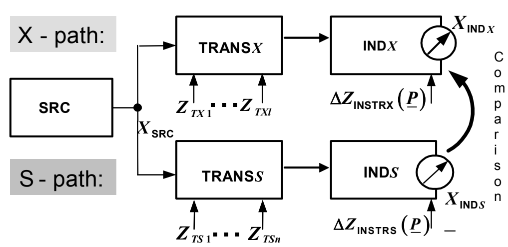

Fig. 10: Generic structure of the cause-and-effect relationship of a calibration employing the *direct comparison method* of two indicating measuring instruments. Symbols: X_{SRC} – quantity provided by a parameter source SRC; TRANSX – transforming unit of the X-path; TRANSS – reference transforming unit; INDX – instrument under test; INDS – indicating standard; $Z_{TX1}, \ldots, Z_{TXl}$ and $Z_{TS1}, \ldots, Z_{TSn}$ – additional input quantities; $\Delta Z_{INSTRX}(\underline{P})$ – instrumental error of the instrument under test (measurand) depending on the calibration conditions $\underline{P}$; $\Delta Z_{INSTRS}(\underline{P})$ – instrumental error of the standard used; X_{INDX} – indication of the instrument under test; X_{INDS} – indication of the standard

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