

CYLINDRICAL AND SPHERICAL COORDINATE TRANSFORMATIONS IN ERROR BUDGETS

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Abstract

Error budgets are powerful tools for analyzing the uncertainty of positioning systems within machinery. A typical error budget employs homogeneous transformation matrices between Cartesian coordinate systems located along multiple pathways through the machine. Although Cartesian coordinates are appropriate for machines that employ linear motion, machines with rotary or spherical joints are better modeled with cylindrical or spherical coordinate systems. Employing Cartesian transformations in such cases may result in overstatement or understatement of the true error of the machine, particularly if second and higher order error terms are neglected. Instead, error terms must be presented as independent radial and angular errors, then converted to errors in a Cartesian coordinate system.

Keywords: error budget, error modeling, coordinate transformations, cylindrical coordinates, spherical coordinates

Introduction

The purposes of an error budget are to predict the uncertainty of positioning systems within machinery and to allocate error specifications to machine components. The error budget ensures that a machine design meets its specifications without being overbuilt. For modern machinery, where nanometer-level positioning uncertainties are expected, second-order effects must often be included within an error budget. Second order effects are also not negligible when one tolerance value is an order of magnitude or more larger than other tolerances in the same machine. Squared values of the larger tolerance may still be comparable in magnitude to the first order terms of the other tolerances.

A typical error budget combines a kinematic model of the machine with error estimates for machine components. The widely adopted approach summarized by Slocum [1] formulates a kinematic model using Cartesian coordinate systems and homogenous transformation matrices for translations, rotations, or both. At the point where the “tool” contacts the “workpiece”, a sensitive direction that critically affects cutting or measurement accuracy is defined, and insensitive directions are defined in directions orthogonal to the sensitive direction. Errors in the non-sensitive direction may have negligible effects and are often ignored.

Although the relationship between the sensitive and insensitive directions can be described by an orthogonal, Cartesian coordinate system, it is sometimes more accurate to express the sensitive and insensitive directions in cylindrical or spherical coordinates. Such is the case for applications with rotating sensitive directions like boring, where the sensitive direction coincides with the radial coordinate and the insensitive direction coincides with the angular coordinate. In such cases it is less appropriate to state that a particular Cartesian direction is insensitive, since errors are actually defined as deviations from a nominal radial value only.

This paper demonstrates that the approach of applying Cartesian coordinates and error values to each transformation may result in either an overstatement or understatement of error. As a consequence, machines may be overbuilt or emphasis may be applied to minimizing what amount to only secondary sources of error.

This paper introduces another approach where the connections between machine components dictate whether the use of Cartesian, cylindrical, or spherical coordinate systems is most appropriate. In this method, an important distinction is made between the mere rotation and transformation of Cartesian coordinate systems and the conversion from Cartesian to cylindrical or spherical coordinates. Errors for a cylindrical surface are evaluated in terms

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of independent r , θ , and z values, rather than assuming independent error terms in x , y , and z . Similarly, errors in spherical coordinates are evaluated by examining independent r , θ , and ϕ error values.

The practice of always assuming a Cartesian coordinate system to represent sensitive and insensitive directions for a tool and workpiece can easily mask the interdependence of the errors, particularly if a Monte Carlo approach is used to determine the errors of the complete system. An example of the results from such a Monte Carlo simulation help illustrate the improvements offered by this new approach.

1. Cylindrical and Spherical Coordinate Transformations

Many machines contain connections or joints that are best modeled with cylindrical coordinates. Polishing or grinding wheels, cylindrical boring or turning machines, and many other designs have rotating elements constrained by spindles that establish axes-of-rotation. In the case where the axis-of-rotation is nominally aligned with the z axis, the transformations from cylindrical coordinates (r , θ , and z) to Cartesian coordinates (x , y , and z) are given by Eq. (1). These coordinates are defined in Fig. 1.

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= z \end{aligned} \tag{1}$$

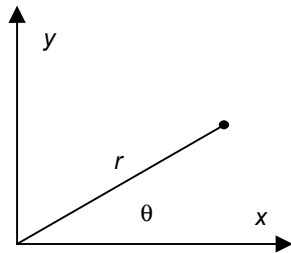


Fig. 1: Definition of polar coordinates.

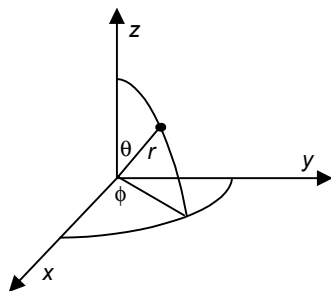


Fig. 2: Definition of spherical coordinates.

Some machines incorporate two rotational degrees of freedom located at the same point, indicating that spherical coordinates are the most suitable. The usual equations for conversion between r , θ , and ϕ are given by Eq. (2). The definitions of these angles are shown graphically in Fig. 2.

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned} \tag{2}$$

2. Cylindrical Examples: Boring and Turning

Consider the production of concave and convex cylindrical surfaces produced by turning or boring as illustrated in Fig. 3 and Fig. 4, respectively. For turning operations such as Fig. 3, where the outside surface is being machined, it is apparent that errors in the angular position θ of the tool produce errors on the surface.

For Fig. 4, where the inside surface is being machined and the center of the tool is nominally coaxial with the axis-of-rotation of the workpiece, errors in the angular position θ of the tool should have no effect. Applying a Cartesian coordinate system to either geometry causes the distinction between these two cases to be obscured. In the coordinate system of the cylinder the sensitive direction is r , and the insensitive directions are θ and z . It is these variables, not x , y , and z , that are the truly independent variables indicating the position of the tool.

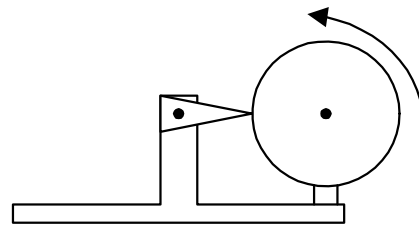


Fig. 3: Turning operation with a fixed tool and rotating workpiece.

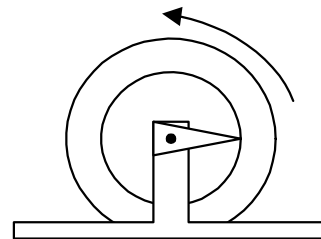


Fig. 4: Boring operation with a fixed tool and rotating workpiece.

For Fig. 4, the tool holder may be designed to provide a highly accurate r value, with less accuracy in its θ positioning. If the conversion from r and θ values to x and y values include second order terms, with x as the sensitive direction, errors in θ may appear to adversely affect the sensitive direction x . This is not a problem with the design but rather with the use of Cartesian coordinates for sensitive and insensitive directions.

With many machines, errors in the radial direction Δr and angular direction $\Delta\theta$ may be independent; encoder resolution, for instance, should be represented as a contributor to $\Delta\theta$ and independent of Δr . Depending on the order of the error model, error in the x direction (Δx) and y direction (Δy) may be statistically correlated even though the errors $\Delta\theta$ and Δr are independent. This correlation, and whether the system error is overstated or understated, depends on the geometry of the machine as shown in subsequent sections.

3. First Order Cylindrical Example

The simplest example to demonstrate this correlation is to apply first order coordinate transformations to the tool in Fig. 4. We assume a finite radius r , a nominal value of $\theta=0$, and an angular error of $\Delta\theta$. Errors in the alignment of the two axes-of-rotation, and any errors in the out-of-plane (z) direction are ignored. First order, small angle approximations are given by Eq. (3).

$$\begin{aligned}\sin \Delta\theta &\approx \Delta\theta \\ \cos \Delta\theta &\approx 1\end{aligned}\quad (3)$$

For this first order approximation the rotation error matrix is shown in Eq. (4), and the translation matrix is shown in Eq. (5). These two matrices are combined to give the give the x and y coordinates as shown in Eq. (6). Second order terms are dropped, to maintain consistency within this approach.

$$T_{1-ord} = \begin{bmatrix} 1 & \Delta\theta & 0 & 0 \\ \Delta\theta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

$$T_{trans} = \begin{bmatrix} 1 & 0 & 0 & r + \Delta r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5)$$

$$T_{1-ord} T_{trans} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r + \Delta r \\ r\Delta\theta \\ 0 \\ 1 \end{bmatrix} \quad (6)$$

From these results it is apparent that the errors Δx and Δy are related to the original, independent variables Δr and $\Delta\theta$, as shown in Eq. (7). These two equations are written in matrix form to highlight the fact that, since the errors Δx and Δy are orthogonal combinations of Δr and $r\Delta\theta$, they are linearly independent. From a purely linear algebra standpoint, they are orthogonal because the dot product of the two row vectors of the matrix in Eq. (7) is zero.

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \Delta r \\ r\Delta\theta \end{bmatrix} \quad (7)$$

4. Second Order Cylindrical Example

The next step to check for correlation is to apply second order approximations to the coordinate transformations of Fig. 4. We once again assume a finite radius r , a nominal value of $\theta=0$, and an angular error of $\Delta\theta$. The second order small angle approximations are given by Eq. (8).

$$\begin{aligned}\sin \Delta\theta &\approx \Delta\theta \\ \cos \Delta\theta &\approx 1 - \Delta\theta^2/2\end{aligned}\quad (8)$$

Using these second order approximations, the rotation error matrix is shown in Eq. (9). The translation matrix is unchanged from the previous example, and so Eq. (5) can be used. These two matrices are combined to give the give the x and y coordinates shown in Eq. (10). Third order and higher terms are dropped in this case.

$$T_{2-ord} = \begin{bmatrix} 1 - \Delta\theta^2/2 & \Delta\theta & 0 & 0 \\ \Delta\theta & 1 - \Delta\theta^2/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

$$T_{2-ord} T_{trans} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r + \Delta r - r\Delta\theta^2/2 \\ r\Delta\theta + \Delta r\Delta\theta \\ 0 \\ 1 \end{bmatrix} \quad (10)$$

From these results it is apparent that the errors Δx and Δy are related to the original, independent variables Δr and $\Delta\theta$, as shown in Eq. (11). Now the errors Δx and Δy are nonlinear combinations of Δr and $\Delta\theta$, and are no longer independent.

$$\begin{aligned}\Delta x &= \Delta r - r\Delta\theta^2/2 \\ \Delta y &= r\Delta\theta + \Delta r\Delta\theta\end{aligned}\quad (11)$$

To show the correlation between Δx and Δy graphically, three different sets of random errors are generated for the tool shown in Fig. 3 or 4. For this data the radius r of the tool is set to 40 cm, the error Δr is ± 0.01 cm, the angle θ is 0, and the angular error $\Delta\theta$ is ± 0.05 radians.

Physical measurement of the errors Δx and Δy would reveal a range of -0.06 to $+0.01$ cm in x , and ± 2 cm in y . Figure 5 shows a uniform distribution of Δx and Δy within these ranges. Each point plotted represents a possible location of the toolpoint relative to the ideal position of $\{0, 0\}$.

An improved method of modeling the errors is to measure Δr and $\Delta\theta$, then use the first order approximation from Eq. (7) to convert these errors to Δx and Δy coordinates. Figure 6 shows uniformly distributed Δr and $\Delta\theta$ variables. Since the first order approximation maintains the independence of the variables, the distribution is still uncorrelated but in a much smaller range.

The third and final method of modeling the errors is to utilize the second order approximations for Δx and Δy , found in Eq. (11). With the same ranges on Δr and $\Delta\theta$, Fig. 7 reveals how the correlation affects the randomly distributed original error variables.

A key question in this study is whether the omission of the second order terms results in an overstatement of error or an understatement of error. Unfortunately, the direction of the effect depends on the unique geometry of the problem to be analyzed. The two machines in Fig. 3 and Fig. 4 are contrasting examples of this effect.

For the convex cylinder of Fig. 3, the data points in Fig. 7 fall further from the ideal surface than the data points in Fig. 6. Neglecting the second order terms for this machine would therefore result in an understatement of error. For Fig. 4, the points in Fig. 7 more closely approximate the interior of the cylinder, therefore the omission of the second order terms results in an overstatement of error. In both cases, the assumption of independent Δx and Δy values as shown in Fig. 5 results in the largest overstatement of error, and should never be applied.

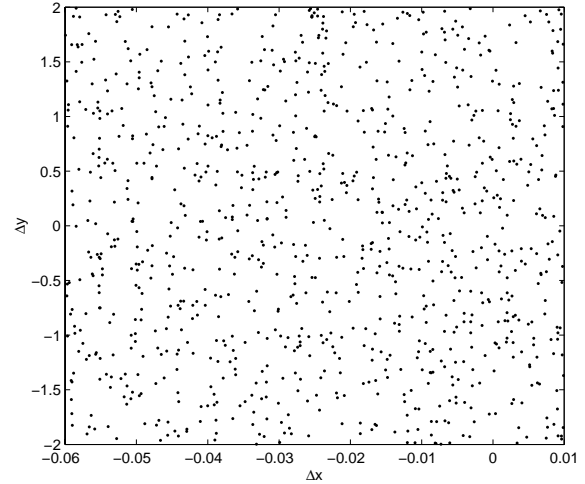


Fig. 5: Example of uncorrelated Δx and Δy .

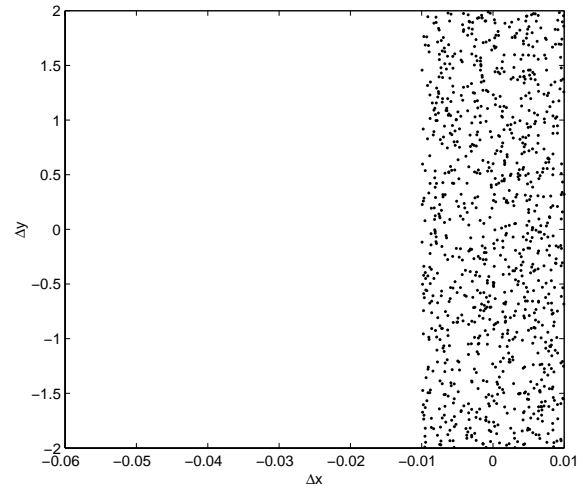


Fig. 6: Example of uncorrelated Δx and Δy , generated from first order approx. for Δr and $\Delta\theta$.

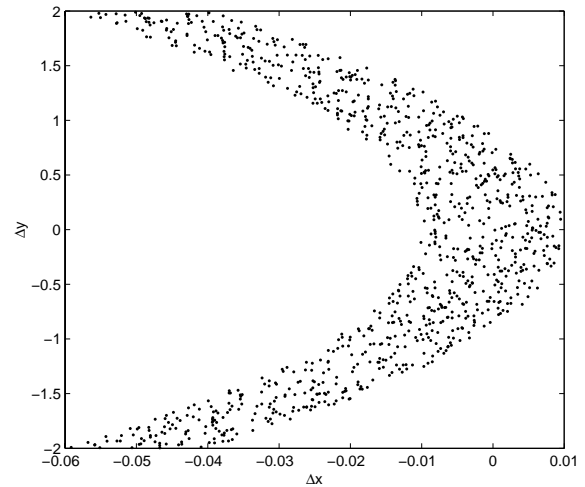


Fig. 7: Example of correlated Δx and Δy , generated from second order approximations for Δr and $\Delta\theta$.

5. Generalized First Order Cylindrical Results

The previous example was somewhat specific in that the machine tool was positioned horizontally, i.e. the tool was not rotated through a finite angle θ . To determine the first and second order equations for Δx and Δy more generally, the transformation in Eq. (12) must be applied. The first order approximations for these angles are shown in Eq. (13).

$$T_{rot} = \begin{bmatrix} \cos(\theta + \Delta\theta) & -\sin(\theta + \Delta\theta) & 0 & 0 \\ \sin(\theta + \Delta\theta) & \cos(\theta + \Delta\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (12)$$

$$\begin{aligned} \sin(\theta + \Delta\theta) &\approx \sin \theta + \Delta\theta \cos \theta \\ \cos(\theta + \Delta\theta) &\approx \cos \theta - \Delta\theta \sin \theta \end{aligned} \quad (13)$$

Multiplying this rotation matrix with the translation matrix of Eq. (5) produces a result that can be written as shown in Eq. (14).

$$T_{rot} T_{trans} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \\ 0 \\ 1 \end{bmatrix} \quad (14)$$

The x , Δx , y , and Δy results can be written once again in matrix form, as shown in Eq. (15). Since the dot product of the two rows of the matrix in Eq. (15) is zero, the vectors Δx and Δy are shown to be orthogonal.

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} \Delta r \\ r\Delta\theta \end{bmatrix} \quad (15)$$

6. Generalized Second Order Cylindrical Results

The second order, general solution is more complex. The cosine and sine approximations, to second order, are shown in Eq. (16). The resulting equations for Δx and Δy are shown in Eqs. (17) and (18). These are neither linear nor orthogonal.

$$\begin{aligned} \sin(\theta + \Delta\theta) &\approx \sin \theta + \Delta\theta \cos \theta - \frac{\Delta\theta^2}{2} \sin \theta \\ \cos(\theta + \Delta\theta) &\approx \cos \theta - \Delta\theta \sin \theta - \frac{\Delta\theta^2}{2} \cos \theta \end{aligned} \quad (16)$$

$$\begin{aligned} \Delta x &= \Delta r \cos \theta - r\Delta\theta \sin \theta \\ &\quad - \Delta r \Delta\theta \sin \theta - \frac{r\Delta\theta^2}{2} \cos \theta \end{aligned} \quad (17)$$

$$\begin{aligned} \Delta y &= \Delta r \sin \theta + r\Delta\theta \cos \theta \\ &\quad + \Delta r \Delta\theta \cos \theta - \frac{r\Delta\theta^2}{2} \sin \theta \end{aligned} \quad (18)$$

7. Generalized First Order Spherical Results

The same technique applied to two-dimensional cylindrical transformations can be applied to the general case of spherical coordinates as defined in Fig. 2. The rotations to convert from cylindrical coordinates r , θ , and ϕ , to Cartesian coordinates x , y , and z , are shown in Eqs. (19) through (21). This example requires two rotations and a translation, to be multiplied as shown in Eq. (22).

$$T_1 = \begin{bmatrix} \cos(\phi + \Delta\phi) & -\sin(\phi + \Delta\phi) & 0 & 0 \\ \sin(\phi + \Delta\phi) & \cos(\phi + \Delta\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (19)$$

$$T_2 = \begin{bmatrix} \cos(\theta + \Delta\theta) & 0 & -\sin(\theta + \Delta\theta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta + \Delta\theta) & 0 & \cos(\theta + \Delta\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (20)$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r + \Delta r \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (21)$$

$$T_1 T_2 T_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \\ z + \Delta z \\ 1 \end{bmatrix} \quad (22)$$

For the first order approximation of sine and cosine, once again Eq. (13) is used. Second order and higher terms are discarded, with the resulting error terms Δx , Δy , and Δz as shown in Eq. (23).

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \theta \sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \sin \theta \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta r \\ r\Delta\theta \\ r\Delta\phi \end{bmatrix} \quad (23)$$

As before, the rows of the matrix in Eq. (23) can be considered as three separate vectors. Unlike the cylindrical example, however, these vectors are not orthogonal to one another. In particular the dot product of Δx and Δy is nonzero, indicating a correlation between these two variables.

This correlation indicates that, for a machine with independent errors in Δr , $\Delta\theta$, and $\Delta\phi$, the related Cartesian errors Δx , Δy , and Δz will always be interdependent. Whether this interdependence results in an understatement or an overstatement of error again depends on the particular geometry of the machine.

8. Generalized Second Order Spherical Results

As in the case of cylindrical transformations, the most accurate calculation of the Cartesian errors Δx , Δy , and Δz must include second order error terms in Δr , $\Delta\theta$, and $\Delta\phi$. The rotation matrices of Eqs. (19) through (21) are again applied, but the second order approximations for sine and cosine are taken from Eq. (16). Terms higher than second order are dropped, and the resulting equations for the Cartesian errors are shown in Eqs. (24) through (26).

$$\begin{aligned} \Delta x = & \left(\Delta r - \frac{r}{2} \Delta\phi^2 - \frac{r}{2} \Delta\theta^2 \right) \sin\theta \cos\phi + \\ & (r\Delta\theta\Delta\phi) \cos\theta \sin\phi - (r\Delta\phi + \Delta r\Delta\phi) \sin\theta \sin\phi \\ & + (r\Delta\theta + \Delta r\Delta\theta) \cos\theta \cos\phi \end{aligned} \quad (24)$$

$$\begin{aligned} \Delta y = & \left(\Delta r - \frac{r}{2} \Delta\phi^2 - \frac{r}{2} \Delta\theta^2 \right) \sin\theta \sin\phi + \\ & (r\Delta\theta\Delta\phi) \cos\theta \cos\phi + (r\Delta\phi + \Delta r\Delta\phi) \sin\theta \cos\phi \\ & + (r\Delta\theta + \Delta r\Delta\theta) \cos\theta \sin\phi \end{aligned} \quad (25)$$

$$\Delta z = \left(\Delta r - \frac{r}{2} \Delta\theta^2 \right) \cos\theta - (r\Delta\theta + \Delta r\Delta\theta) \sin\theta \quad (26)$$

Although these equations are admittedly convoluted, terms are grouped in an attempt to gauge the relative effects of different terms. If the radius r and the angular errors are large, second order terms such as $r\Delta\theta\Delta\phi$ may have a significant effect. Conversely, second order terms that are deemed negligible can be selectively eliminated from Eq. (24) through (26).

10. References

- [1] Slocum, A. H., *Precision Machine Design*, Society of Manufacturing Engineers, Dearborn, MI (1992).

9. Conclusions and Future Work

We have shown that the errors for many machines components are best described using cylindrical or spherical coordinates. As these errors are converted to Cartesian coordinates, care must be taken in evaluating the dependence or independence of the resulting Cartesian errors.

For the example of cylindrical machining with a single point tool, it was shown that the first order approximation resulted in independent variables in the Cartesian coordinate system. The second order approximation, however, showed a strong correlation between the errors in the Cartesian coordinates. It was also shown that overlooking this correlation may result in an overstatement or understatement of the machine error, depending on the exact geometry of the machine.

For the general case of spherical coordinates, the conversion equations between spherical errors terms and Cartesian error terms are shown. Even in the case of the first order equations, correlation is found between some of the Cartesian variables. This indicates that the errors for a machine with independent spherical degrees of freedom should first be modeled as spherical coordinate errors. The errors in the Cartesian coordinate system must then be calculated from the spherical errors.

It is assumed throughout the cylindrical analysis of Fig. 4 that the axis of rotation of the tool was coincident with the axis of rotation of the workpiece. In a complete design analysis, errors in positioning will affect this alignment, and additional transformation matrices are necessary. Depending on the magnitude of the translations and the rotation errors, second order terms may or may not be necessary throughout the model.

It has also been brought to the attention of the authors that it is often necessary to solve the reverse problem of the transformations discussed here. For example, it is often necessary to determine what effect errors in Δx , Δy , and Δz will have on positioning in spherical coordinates. Unfortunately, no simple transformation matrices exist to produce a matrix of Δr , $\Delta\theta$, and $\Delta\phi$ values from Δx , Δy , and Δz . General equations to study the effect of these errors may be considered at a future time.