Industrial Applications of Uncertainty Analysis

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Abstract:

Uncertainty or tolerance analysis is a powerful technique for addressing many problems facing the practicing engineer. This paper reviews several of the common uses of this tool and introduces some novel applications.

The paper begins with derivation of "Law of Error Propagation" and follows the developments presented earlier.[1] In this development, explicit forms for the outcome of a process and its statistics are developed.

These results are then applied to a number of common industrial applications. The applications include, error budgeting and requirements derivation. Included in the discussion of the derivations of requirements is a discussion of the need to include margin and contingency explicitly into the analysis.

Next, an expression for the repeatability of a process is derived. This analysis is then used to identify the general conditions that lead to a repeatable process.

Combining cost data or modeling, error budgets are extended into tools that can be used in Cost as an Independent Variable (CAIV) modeling and analysis. The combined model can then be used to determine cost as function of tolerance and the marginal cost of repeatability.

Introduction

This paper derives the Law Of Error Propagation and explores its uses in determining the outcome and distribution of a process in the presence of variations. The explicit purpose of this paper is tutorial in nature, specifically to fill a void in the existing literature on the origins of this important result. In starting from first principles, significant insight can be gleaned on the wide applicability and utility of this powerful Law.

The derivation begins with a general mathematical model of a measurement or process. The sensitivities of the expected outcome of the process to variances in the individual parameters of the problem are derived. The analysis continues and gives expressions for the expectation value (most likely outcome) of the process, μ, in the presence of the variances. A second product of this derivation is the variance of the distribution about μ, generally denoted, σ^2. The derivation

[1] J.W. Arenberg, "On the Origins of the Law of Error Propagation and its Uses ", American Society for Precision Engineering Summer Topical Meeting on Tolerancing, Charlotte, N.C., July 2002.

clearly identifies the assumptions leading to the formula for error propagation, and under what conditions the traditional rss formula is applicable. It will be shown that the traditional rss expression is a special case of the more general Law of Error Propagation, where all parameters of the problem are not independent. The paper will also clearly identifies which statistic is to be combined in an rss fashion.

Outcome of a Process

This section derives the expression for the mean and variance of a process in the presence of variations about an ideal or desired set of parameters. These outcomes, mainly the expression for the variance are well known as the Law of Error Propagation.

Consider a process or procedure of many variables that has an outcome z. The outcome can be the result of a manufacturing process, measurement or even a high level system performance parameter. The outcome, z, is called the measurand, and can be represented as the mathematical process, f, operating on the parameters or variables of the problem denoted, $w_1, w_2 \ldots w_N$ and is written

$$z = f(w_1, w_2 \ldots, w_N) \tag{1}$$

Let the optimum, desired or "true" value of z be denoted, μ_z the set of w_i that give the value of μ_z, are denoted μ_i. Thus, the optimum process is written

$$\mu_z = f(\mu_1, \mu_2 \ldots, \mu_N). \tag{2}$$

Each w_i is has a set of likely values according to its own probability distribution, $p_i(w_i)$. If the probability densities of the w_i about μ_i are known, it is possible to calculate directly the distribution of likely outcomes of z. Assuming that the probability distributions are functions of only one w_i and are independent, the probability density of an outcome z is given by

$$p(z) = f(w_1, w_2, \ldots w_N) \prod_{i=1}^{N} p(w_i) \tag{3}$$

Most often (3) is evaluated using Monte Carlo techniques. Armed with the knowledge of the probability density, all of the moments of z can be calculated directly. A frequently used process is to fit the outcome of the Monte Carlo calculation (usually a histogram) To a known distribution for further analysis.[2]

A useful application is estimating the probability of meeting a specific requirement, denoted R (where higher is a better performance) . This probability is written

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[2] A recent example is given in “An intermodule alignment budget for the Generation X telescope”, Jonathan Arenberg and Christopher Lau, in SPIE 5488 (2004). To be published.

$$\Pr(z \geq R) = \int_{R}^{\infty} p(z)dz \,. \tag{4}$$

If p(z) has been parameterized in terms of μ_i and σ_i then the probability of meeting a given performance can be expressed in terms of the design point and tolerance set.

In some cases, it is sufficient to only calculate, σ_z. In these situations, small deviations from the μ_i and their impact on μ_z are calculated from a Taylor Series expansion about the μ_i which is

$$z = \mu_z + \sum_{i=1}^{N} \frac{\partial f}{\partial w_i}\left(w_i - \mu_i\right) + O\left(\left(w_i - \mu_i\right)^2\right). \tag{5}$$

Ignoring terms higher than first order in $\left(w_i - \mu_i\right)$ gives

$$z = \mu_z + \sum_{i=1}^{N} \frac{\partial f}{\partial w_i}\left(w_i - \mu_i\right). \tag{6}$$

Equation (6) is the first major result of this derivation. It gives the value of the measurand in the case where the w_i are not equal to the μ_i. Next, μ_z is subtracted from both sides of (6) to give

$$z - \mu_z = \sum_{i=1}^{N} \frac{\partial f}{\partial w_i}\left(w_i - \mu_i\right). \tag{7}$$

The term on the left hand side is the deviation from the ideal value, μ_z. Squaring (7) gives

$$\left(z - \mu_z\right)^2 = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial w_i}\right)^2 \left(w_i - \mu_i\right)^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\partial f}{\partial w_i}\right)\left(\frac{\partial f}{\partial w_j}\right)\left(w_i - \mu_i\right)\left(w_j - \mu_j\right). \tag{8}$$

Next, the expectation value, E(x)[3], is taken of both sides of (8)

$$E\left[\left(z - \mu_z\right)^2\right] = E\left[\sum_{i=1}^{N}\left(\frac{\partial f}{\partial w_i}\right)^2 \left(w_i - \mu_i\right)^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\partial f}{\partial w_i}\right)\left(\frac{\partial f}{\partial w_j}\right)\left(w_i - \mu_i\right)\left(w_j - \mu_j\right)\right]. \tag{9}$$

Since the expectation value of a sum is the sum of the expectation values, the right hand side of (9) is written

[3] The expectation value E(x) is given by the expression $E\left(x\right) = \int_{-\infty}^{\infty} xp\left(x\right)dx$, where p(x) is the probability distribution of x.

$$\sigma_z^2 = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial w_i}\right)^2 E\left[\left(w_i - \mu_i\right)^2\right] + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\partial f}{\partial w_i}\right)\left(\frac{\partial f}{\partial w_j}\right) E\left[\left(w_i - \mu_i\right)\left(w_j - \mu_j\right)\right]. \qquad (10)$$

The change in the left hand side of (10) is made since the variance of z, is the expectation value of, $z - \mu_z$, namely, $\sigma_z^2 = E(z_i - \mu_z)$. Equation (10) reduces to

$$\sigma^2 = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial w_i}\right)^2 \sigma_i^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\partial f}{\partial w_i}\right)\left(\frac{\partial f}{\partial w_j}\right)\sigma_{ij} \qquad (11)$$

σ_{ij} is called the covariance of i and j. The covariance is given by

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} \qquad (12)$$

where ρ_{ij} is the correlation coefficient. So,

$$\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}. \qquad (13)$$

Substitution of (13) into (11) gives

$$\sigma_z^2 = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial w_i}\right)^2 \sigma_i^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\partial f}{\partial w_i}\right)\left(\frac{\partial f}{\partial w_j}\right)\rho_{ij}\sigma_i\sigma_j. \qquad (14)$$

Equation (14) is the general expression of the variance in the measurand. No assumptions about the nature of the $p_i(w_i)$ were made and so the result, (14) is quite general and applicable.

For parameters that are independent, the covariance is zero, and (14) becomes

$$\sigma_z^2 = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial w_i}\right)^2 \sigma_i^2. \qquad (15)$$

which is the well known result and the goal of this derivation . Also note that the σ_i^2 are the variances for each variable i. From this derivation is clear that the quantities that are rss'ed are the σ_i and not some other statistic.

Error Budgeting

A chief industrial application of this class of analysis is the error (tolerance or performance budget). This is a device to capture and communicate the design point and more commonly the allowed deviations from the design point. The error budget is the set of deviations from $\vec{\mu}$ that still satisfy requirements. These errors are most usually and easily expressed as the standard deviations, σ_i, around the μ_i. The process of adjusting difficulty and cost among the σ_i is referred to as error budgeting.

Discussion of the error budget will be extended to include a procedure to determine adequate margin and contingency so that a specific yield (probability of success) can be attained.

Repeatability

The repeatability of a process is the probability that a second outcome agrees to within ε of the first. If the this probability is denoted, R(ε) and the two outcomes are Λ_1 and Λ_2, the R(ε) may be written

$$R(\varepsilon) = \Pr\left(\left|\Lambda_1 - \Lambda_2\right| \le \varepsilon\right) = \Pr\left(\Lambda_1\right)\Pr\left(\Lambda_2 \le \left|\Lambda_1 - \varepsilon\right|\right) \ \ \forall \ \ \Lambda_1. \tag{16}$$

The probability density of Λ is p(z) namely, so (16) becomes

$$R(\varepsilon) = \int_0^\infty p\left(\Lambda_1\right) dz_1 \cdot \Pr\left(\Lambda_2 \le \left|\Lambda_1 - \varepsilon\right|\right) \tag{17}$$

which is

$$R(\varepsilon) = \int_0^\infty p\left(\Lambda_1\right) d\Lambda_1 \left\{ \int_{\Lambda_1 - \varepsilon}^{\Lambda_1 + \varepsilon} p(\Lambda)\, d\Lambda \right\}. \tag{18}$$

Equation (18) can be written in equivalent form in terms of the cumulative distribution of the first order statistic $\Psi(x_{(1)})$

$$R(\varepsilon) = \int_0^\infty p\left(\Lambda_1\right)\left\{P\left(\Lambda_1 + \varepsilon\right) - P\left(\Lambda_1 - \varepsilon\right)\right\} d\Lambda_1, \tag{19}$$

where

$$P(\Lambda) = \int_0^\Lambda \psi\left(\Lambda'\right) d\Lambda'. \tag{20}$$

The value of R(ε) will be largest when the term in the braces is large, either via large value of ε, or a tight distribution (steep cumulative function).

Other Applications

As shown above in (4), the probability of meeting a given requirement can be determined for a specific process, design point and tolerance set. If the $\Pr(z \ge R)$ is denoted Q (Q for quality) and cost relationships or models exist for the mi and si and expression for the cost, C of having a given probability of equaling or exceeding a specific outcome can be formulated, $C\left(R,\vec{\mu},\vec{\sigma}\right)$. Q can be plotted against C for various design points and tolerance sets. This analysis allows for direct examination of cost versus quality and what elements in the design and tolerance space are most influential.

Conclusion

This paper has presented several common industrial applications of uncertainty analysis. Extensions to these basic techniques will be discussed in the oral presentation. It is hoped that this paper will contribute to the propagation of these methods and associated formalism.