# MOMENT OF INERTIA MEASUREMENT USING A FIVE-WIRE TORSION PENDULUM AND OPTICAL SENSING

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## INTRODUCTION

Typical moment of inertia measurement devices are capable of $1$ part in $10^3$ accuracy and current state of the art techniques are approaching capabilities of a few parts in $10^4$ [1]. Introduced here is a new method for measuring the moment of inertia using a novel five-wire torsion pendulum design, which shows the prospect of improving on current state of the art.

A moment of inertia measurement apparatus typically attempts to produce a pure rotation about one degree of freedom. The measurements of rotation can have uncertainties when there are significant other degrees of freedom. Bifilar and trifilar pendulums, for example, do not constrain the swinging or lateral translation modes. The five-wire design reduces errors due to tilt and horizontal translational degrees of freedom.

The five-wire torsion pendulum is integrated with optical angular sensing using diffraction grating angular magnification. Optical sensing decouples any vertical motion from the detection sensor signal and the associated angular magnification allows for a high sensitivity measurement.

## MEASUREMENT

### Measurement Apparatus

When designing an apparatus to measure the moment of inertia to a high precision, care must be taken to minimize the extra degrees of freedom in the system. To improve the accuracy of a standard trifilar pendulum, the lateral pendulum modes need to be constrained. In a five-wire pendulum, two additional wires are arranged as shown in figure 1 to minimize rotations about the other two rotational axes. The

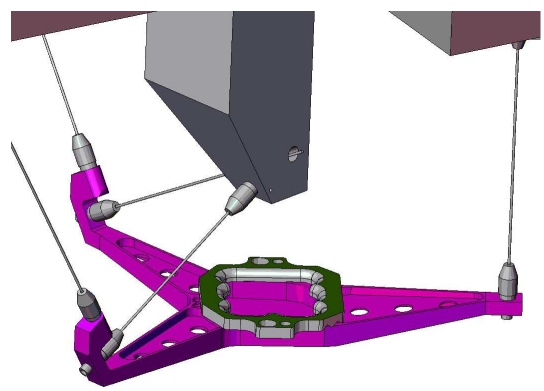

*FIGURE 1. Torsion pendulum platform attach points for the five wires. A kinematic orientation fixture is in the center.*

three attach points on the platform supporting the inertia to be measured are positioned equidistant from the center of rotation. At one attach point, a single vertical wire is used and the other two attach points consist of two wires. The horizontal components of the wires which are splayed out from a single attach point provide horizontal stiffness to prevent pendulum platform swinging motion. The wires are attached to the supporting frame along a line emanating from the rotation center at a point in the plane containing the mass center of the object to be measured. The choice of the attach point locations on the supporting frame ensures a constant curvature at the platform wire mount points from each wire, which coincides with the curvature of the pendulum rotation. The constant curvature constraint is necessary to ensure proper rotation about the vertical axis.

The full moment of inertia tensor contains six independent terms. As such, a moment of inertia measurement device must be capable of determining the moment of inertia about at least six different axes of rotation. A torsion pendulum design is capable of measuring the inertia about one axis. Therefore, the apparatus requires the capability of re-orienting the measurement object to at least six independent directions. The five-wire measurement apparatus utilizes a kinematic fixture which permits changing the orientation of the object with high repeatability. A set of sleds to generate different orientations were constructed. These kinematic orientation fixtures are repeatable to 0.02$^\circ$. The fixtures are mounted to the pendulum platform using a set of v-grooves and ball bearings for a repeatable placement.

## Grating Angular Sensing

The oscillatory motion of a torsion pendulum is described by the relationship between the radius of gyration $R_g$, or the instantaneous moment of inertia about the axis of rotation $I_p$, and the angular frequency $\omega$

$$R_g^2 = \frac{I_p}{m_p} = \frac{k}{\omega^2} \tag{1}$$

where $k$ is the torsion coefficient or stiffness constant of the pendulum and $m_p$ is the total mass of the pendulum. The moment of inertia for the system is therefore determined by measuring the angular frequency $\omega$ and the pendulum mass. The angular displacement of the five-wire pendulum is measured by application of a grating angular sensor [2]. A diffraction grating is attached to the pendulum platform at the center of rotation. Using a laser as a light source, the grating diffraction orders are used for angular sensing as depicted in figure 2. As the pendulum platform rotates, the

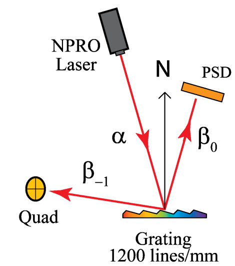


*FIGURE 2. Setup of the grating angular sensor on the five-wire torsion pendulum.*

diffracted beams rotate similar to a plane mirror reflection but are additionally magnified by the diffraction order angle. Any minor translational motion of the pendulum will not be magnified and vertical motion of the pendulum will not affect the diffracted beam. If the pendulum rotates (and hence the attached grating) by an angle of $\delta\alpha$, the resulting incremental rotation angle $\delta\beta_n$ of the diffracted beam for diffraction order $n$ is given by [2]

$$\delta\beta_n = \frac{\cos(\alpha)}{\cos(\beta_n)}\,\delta\alpha \tag{2}$$

where $\alpha$ and $\beta_n$ are the angles from the grating norm for the incident beam and the diffraction angle respectively. This grating angle magnification enhances the measurement sensitivity. The grating angular sensor magnifies the angular displacement of the pendulum but does not magnify any translation.

The diffracted beams are captured by a position sensitive diode (PSD) at the $0$ diffraction order and a quad photo diode at the $-1$ diffraction order. The diffraction order angles of $81^\circ$ and $16.8^\circ$ were chosen to combine the advantages of a high dynamic range sensor with high precision.

## Calibration

As with any measurement device, the apparatus must first be calibrated before measurements can be made. For the five-wire torsion

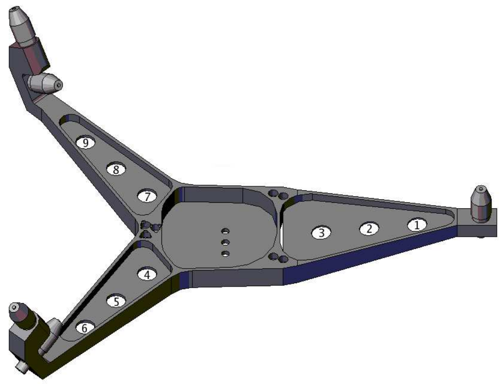

*FIGURE 3. Pendulum Platform. The nine large holes are used for calibration sphere placement.*

pendulum, the moment of inertia for the pendulum platform must be determined. The oscillatory motion of a torsion pendulum is described by the relationship in equation 1. The unknown moment of inertia and torsion coefficient for the pendulum are determined by adding a known moment of inertia to the system. This is achieved by the addition of $n$ calibration spheres to the platform with a known moment of inertia $I_c$, such that the relationship in equation 1 becomes

$$I_p + \sum_{i=1}^{n} I_{c_i} = \left( m_p + \sum_{i=1}^{n} m_i \right) \frac{k}{\omega^2} \quad (3)$$

Each calibration sphere mass $m_i$ with radius $a_i$, is placed in holes at locations $R_i$, determined by a coordinate measuring machine. The use of fixed holes for locating the calibration spheres ensures a high repeatability for positioning and hence a consistent addition to the total system moment of inertia. Since the calibration spheres are not placed at the rotation center of the pendulum platform, the parallel axis theorem is applied and equation 3 becomes

$$I_p + \sum_{i=1}^{n} m_i R_i^2 + \tfrac{2}{5} m_i a_i = \left( m_p + \sum_{i=1}^{n} m_i \right) \frac{k}{\omega^2}$$

The calibration measurement is repeated for various combinations of calibration sphere positions. A least squares fit of the measured data is then performed to obtain the unknown parameters. During the calibration process, the same number of spheres are used to ensure a constant system mass, avoiding a change in the torsion coefficient for the system. In addition, the calibration process is performed with the orientation fixtures in place, such that the only addition to the system after calibration is the object for which the moment of inertia is to be determined.

## PRELIMINARY RESULTS

The pendulum platform contains a small permanent magnet, surrounded by two wire coils, which when energized produces a magnetic field and hence a torque on the platform. By sending a pulse to the coils using a function generator, a repeatable pendulum response is generated. The time history angular position of the platform is recorded from the PSD and Quad photo diode signal. The pendulum oscillatory frequency is easily extracted by data reduction in the time domain. The response to a single disturbance to the pendulum is known to be governed by the equations for damped harmonic motion. A nonlinear curve fit is then applied to the data to experimentally determine the parameters of the pendulum response, including the natural frequency and damping coefficient. Shown in figure 4 is a representative time domain curve and fit to the data obtained during the calibration procedure with the $5^\circ$ orientation fixture in place.

During the calibration process, multiple measurements of the oscillation frequency were performed for each combination of the calibration sphere positions. The individual calibration measurements indicated a repeatability on frequency determination for the time domain curve fit data to be within the standard deviation range of $\sigma = 2 \times 10^{-6}$ to $9 \times 10^{-6}$ Hz for a frequency of around 2.9 Hz, depending on the particular configuration for the calibration spheres. After measuring the frequency for various combinations of calibration sphere locations, a least squares analysis was performed

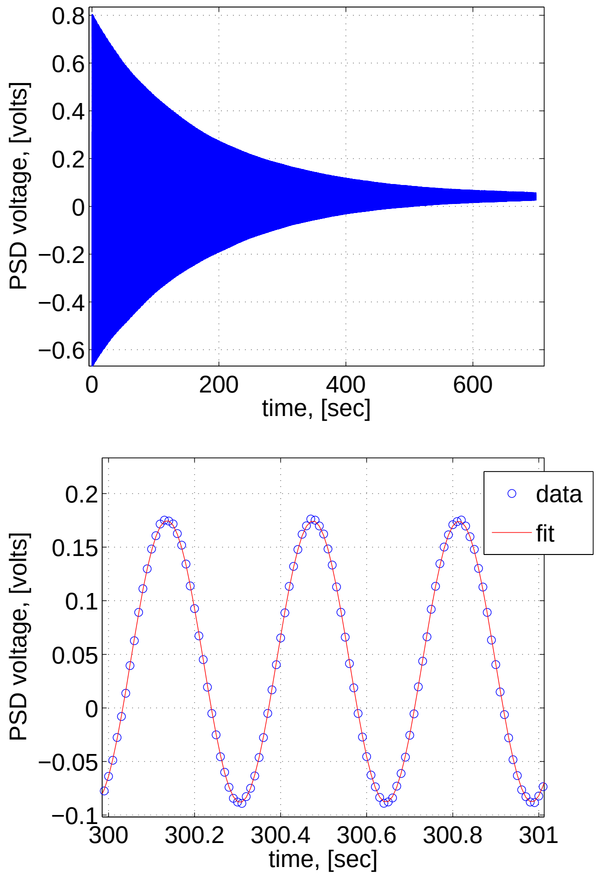


*FIGURE 4. A representative segment of data for the pendulum angular position data and curve fit. The data was obtained with the $5^\circ$ orientation fixture and the three calibration spheres placed in the innermost hole locations.*

to determine the moment of inertia about the rotation axis for the pendulum platform with the $5^\circ$ orientation fixture in place. By including different combinations of calibration measurements in the least squares analysis, a preliminary value for the pendulum platform moment of inertia with the $5^\circ$ orientation fixture in place was found to be 5632 $\mathrm{g \cdot cm^2}$ with a standard deviation of $\sigma = 1.0\,\mathrm{g \cdot cm^2}$ resulting in a repeatable precision on the order of a few parts in $10^4$.

## CONCLUSIONS

The precision of the measured moment of inertia can not be more precise than the uncertainty associated with the calibration of the measurement device and the measurement of the object's mass. Currently only the calibration phase of the measurement phase has been conducted. The preliminary results indicate a repeatable precision on the order of a few parts in $10^4$ for the calibration process, suggesting the measurement apparatus could be capable of near state of the art precision. The current repeatability of the frequency measurements leads us to believe that there is a possibility for improvement of the current state of the art. It should be noted that a full error analysis on the acquired data has not been completed. As such the preliminary results can only give insight into the repeatability of the measurements and not a true representation for the *accuracy* of the measurements.

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