

# Defining the measurand in radius of curvature measurements

Angela D. Davies<sup>a</sup> and Tony L. Schmitz<sup>b</sup>

<sup>a</sup> University of North Carolina at Charlotte, Department of Physics and Optical Science, 9201 University City Blvd., Charlotte, NC 28223-0001

<sup>b</sup> Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL 32611

The radius of curvature of an optical element is one of the dominant parameters that determines optical power. Radius errors can be partially compensated by element respacing, but this is time-consuming and costly and can lead to unwanted wavefront aberrations. For spherical components, the radius is usually defined as the radius of the best-fit sphere over the clear aperture of the lens in a least-squares sense. Many measurement methods are available, including comparison with ‘known radius’ test plates, direct image formation/knife-edge test, astigmatism measurement, spherometers, autocollimator with pentaprism, and shearing interferometers<sup>1,2</sup>, but the optical radius bench generally gives the lowest uncertainty. The radius is measured on an optical bench by using a figure measuring interferometer to identify the location of the test artifact at two critical positions, confocal and cat’s eye. The reflected wavefront matches the wavefront exiting the interferometer at these two locations, resulting in a nulled cavity. A displacement gauge measures the displacement of the part between the two positions, and the distance is nominally the radius of the best-fit sphere to the surface. To carry out a traceable radius measurement, however, all biases must be corrected and all uncertainty sources identified and combined to yield an overall combined standard uncertainty - a formidable task for this deceptively simple measurement.

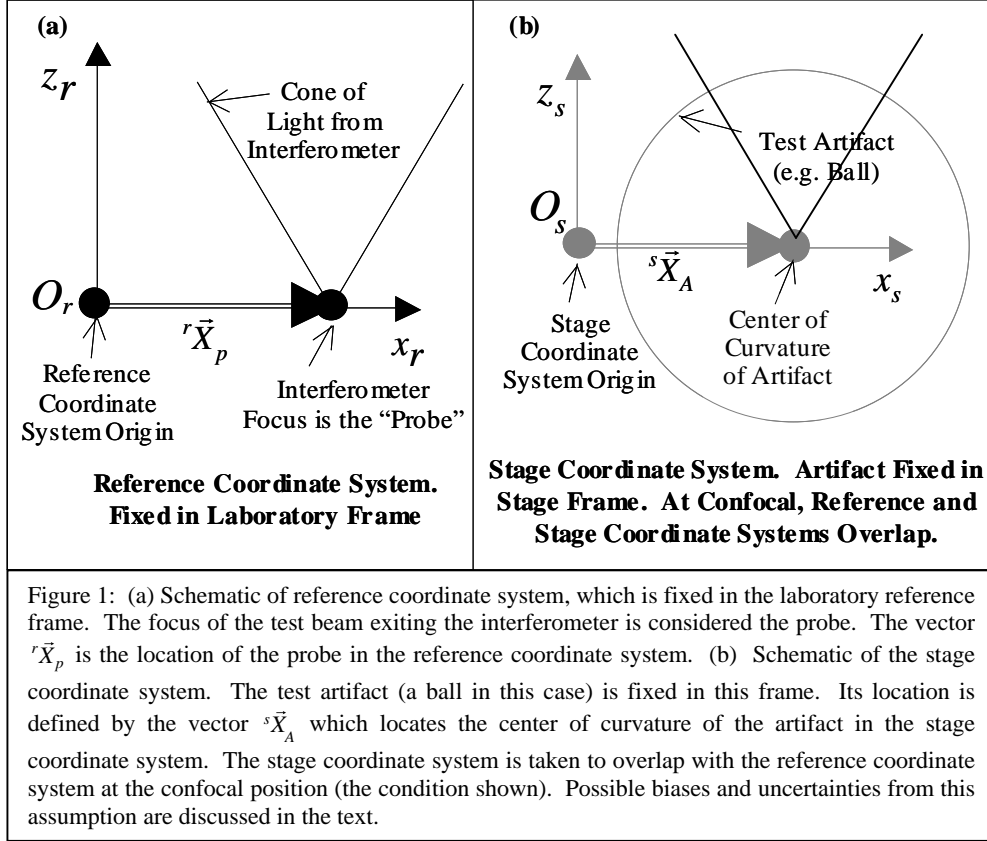
There are many sources of uncertainty and measurement bias, as discussed in the literature<sup>3-5</sup>, such as wavefront aberrations, identification of null at the two locations, figure error corrections, displacement gauge reading, and stage error motions. Error motions are particularly challenging because they are interrelated and convolved with the measurement. Recent work has focused on identifying uncertainty contributions, removing measurement biases, and estimating the uncertainty through a root-sum-of-the-squares (RSS) combination<sup>4,5</sup>. However, a mathematical formalism has yet to be presented which defines the radius as a function of all uncertainty sources. The radius is not a simple linear function, and a mathematical model is needed to ensure that all uncertainty sources are properly combined.

Here we report on a method of mathematically defining the measurand,  $R$ , for optical bench radius measurements that yields a single function that intrinsically corrects for error motions and also allows other uncertainty sources to be represented. The method is based on a homogeneous transformation matrix (HTM) formalism. The formalism is well known in the precision engineering community and is commonly used to analyze error motions in precision machine tools and stages<sup>6</sup>. We use a vector equation to define the radius and an HTM to determine the location of the test artifact after translation from confocal to cat’s eye. To use our results, the test engineer must first characterize the error motions of the radius bench and assess uncertainties, then calculate a corrected estimate of the radius and its uncertainty based on the formalism described here.

In order to correct for stage error motions, the final location of the test artifact after translation must be determined through a coordinate transformation and this requires knowledge of the rigid-body motion of the stage. The uncertainty goal and the extent of the error motions determine the level of sophistication with which the coordinate mapping must be done. One can consider three general categories:

- (i) The error motions are zero on average. Surprisingly, even in this limit, the simple reading of the gauge at cat’s eye compared to confocal is not a good estimate of the radius. Error motions, either positive or negative, always lead to a gauge reading that is less than the actual radius, and this introduces a bias in the measurement, even if the error motions are zero on average<sup>7</sup>. Our formalism intrinsically corrects for this bias.
- (ii) The error motions increase linearly with displacement.
- (iii) The error motions are nonlinear functions of the displacement. This is always the case at some level. This limit must be considered in very demanding uncertainty applications or where the instrument is poorly designed and/or fabricated.

Only the rigid body motion that occurs *during* the displacement is important; consequently, the absolute position of the stage is not critical for categories (i) and (ii), but is for category (iii). The results presented here apply to categories (i) and (ii). Category (iii) requires a more sophisticated treatment.



which is referred to as the reference coordinate system. We define the z-axis of the reference frame as the axis of the displacement gauge. Because we are considering categories (i) and (ii) only, the absolute z-coordinate of the probe or artifact is irrelevant, consequently we set each to zero. The two coordinate systems are shown in Figure 1. The probe vector,  ${}^r\vec{X}_p$ , is defined in the gauge calibration process, which defines the reference z-axis orientation and location. The location of the artifact on the stage is defined at the starting position for the measurement – the confocal position. Because we restrict our analysis to categories (i) and (ii), we can define the stage and reference coordinate systems to exactly overlap at confocal. Consequently the vectors  ${}^r\vec{X}_p$  and  ${}^s\vec{X}_A$  are equal at this point, but only up to the ability to null the interferometer at confocal. This gives us  ${}^s\vec{X}_A = {}^r\vec{X}_p + (dx^{cf}, dy^{cf}, dz^{cf})$ , where the last vector represents nulling errors.

Figure 2 illustrates a radius measurement – translation from the confocal to the cat’s eye position. Current practice is to equate the radius with the gauge reading (assumed zeroed at the starting confocal position). This corresponds to the projection of the center of the artifact onto the z-axis at cat’s eye. As illustrated in the figure with x-axis straightness error in the motion, the gauge reading at cat’s eye ( $d_{ce}$ ) is less than the radius, consequently the measurement is biased. Rather than define the radius in terms of the projection, it can be exactly defined with the appropriate vectors, as shown in the figure. If the location of the artifact at cat’s eye is known in the reference frame, the vector defining the radius is given by  $\vec{R} = {}^r\vec{X}_p - {}^r\vec{X}_A^{ce}$ , and the magnitude of  $\vec{R}$  is given by

$$R^2 = |\vec{R}|^2 = \left| {}^r\vec{X}_p - {}^r\vec{X}_A^{ce} \right|^2.$$

To determine the location of the artifact at cat’s eye, the rigid body motion of the stage must be known. With this knowledge we can map the coordinates of the artifact back to the reference coordinate system by using an HTM<sup>7</sup>. An HTM is a 4 x 4 matrix that captures the rotation and/or translation of one coordinate system relative to another and allows a vector in one system to be mapped to the other. In general, matrices that represent rotation about orthogonal axes do not commute, but they do in the limit of small rotations<sup>7</sup>. An assumption of small rotations is reasonable for a well-designed radius bench, and in this limit, a general HTM for a nominal displacement of  $d$  along the z-axis is given by<sup>7</sup>

The method requires the definition of two coordinate systems and two critical vectors. The two systems are a system fixed to the stage and a one fixed to the focus of light exiting the interferometer that completes the metrology loop. The test artifact (e.g. a ball) is rigidly attached to the stage and there is a unique vector that defines its location,  ${}^s\vec{X}_A$ , taken to be the center of curvature. The location of the beam focus is the other critical vector,  ${}^r\vec{X}_p$ .

The beam focus, or probe, is fixed in the laboratory frame,

$${}^rT_s = \begin{bmatrix} 1 & -\varepsilon_z & \varepsilon_y & \delta x \\ \varepsilon_z & 1 & -\varepsilon_x & \delta y \\ -\varepsilon_y & \varepsilon_x & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

where the angular error motions are captured by the top 3x3 part of the matrix, and the translation errors by the last column. The 4<sup>th</sup> row allows the rotation and translation to be incorporated into the same matrix and can be used to modify scaling. We can apply the HTM to our problem by populating  ${}^rT_s$  with the gauge reading at cat's eye,  $d = d_{ce}$ , and the estimated error motions of the stage during the translation. Multiplying the error motion HTM and the vector that defines the artifact location in the stage frame determines the artifact location in the reference frame after the motion, namely  ${}^r\vec{X}_A^{ce} = {}^rT_s \cdot {}^s\vec{X}_A$ . After substituting into our vector equation defining  $\vec{R}$ , we have

$$\vec{R} = {}^r\vec{X}_p - {}^rT_s \begin{bmatrix} r x_p + dx^{cf} \\ r y_p + dy^{cf} \\ dz^{cf} \\ 1 \end{bmatrix}. \quad (2)$$

With substitution of Equation 1 for the HTM and following the multiplication, we can write out the components of  $\vec{R}$ . We need to calculate the magnitude of the vector  $\vec{R}$ , which involves squaring and summing the three coordinates. Grouping similar terms we end up with

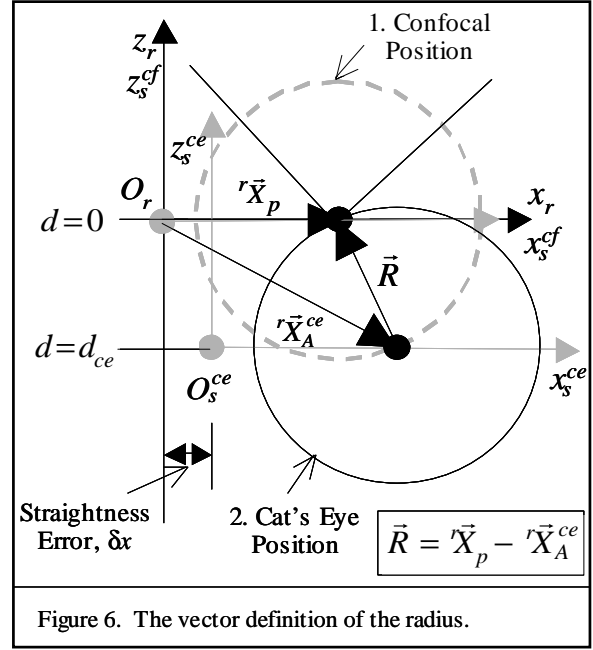
$$\begin{aligned} R^2 = & d_{ce}^2 + \delta x^2 + \delta y^2 + (dx^{cf})^2 + (dy^{cf})^2 + (dz^{cf})^2 + \\ & \varepsilon_x^2 \left( (r y_p)^2 + 2 r y_p dy^{cf} + (dy^{cf})^2 \right) + \varepsilon_y^2 \left( (r x_p)^2 + 2 r x_p dx^{cf} + (dx^{cf})^2 \right) + \varepsilon_y^2 (dz^{cf})^2 + \varepsilon_x^2 (dz^{cf})^2 + \\ & \varepsilon_z^2 \left( (r x_p)^2 + 2 r x_p dx^{cf} + (dx^{cf})^2 \right) + \varepsilon_z^2 \left( (r y_p)^2 + 2 r y_p dy^{cf} + (dy^{cf})^2 \right) + \\ & 2 dx^{cf} \delta x + 2 dy^{cf} \delta y + 2 dz^{cf} d_{ce} + \\ & 2 \varepsilon_z \delta y r x_p - 2 \varepsilon_z \delta x r y_p + 2 \varepsilon_x r y_p d_{ce} - 2 \varepsilon_y r x_p d_{ce} + 2 dy^{cf} \varepsilon_x d_{ce} - 2 dx^{cf} \varepsilon_y d_{ce} + \\ & -2 dx^{cf} \varepsilon_z (r y_p - \delta y) + 2 dy^{cf} \varepsilon_z (r x_p - \delta x) + 2 dz^{cf} \varepsilon_x (r y_p - \delta y) - 2 dz^{cf} \varepsilon_y (r x_p - \delta x) + \\ & -2 \varepsilon_x \varepsilon_y (r x_p r y_p + r x_p dy^{cf} + r y_p dx^{cf} + dx^{cf} dy^{cf}) - 2 \varepsilon_x \varepsilon_z (r x_p dz^{cf} + dx^{cf} dz^{cf}) - 2 \varepsilon_y \varepsilon_z (r y_p dz^{cf} + dy^{cf} dz^{cf}) \end{aligned} \quad (3)$$

This equation is used to estimate the radius from a measurement by evaluating the expression with the expectation values of each term. Our measurand,  $R$ , is equal to the square root of this expectation value, namely  $R = \sqrt{\langle R^2 \rangle}$ . We have assumed that all parameters are uncorrelated.

At first Equation 3 appears daunting, but most of these terms are zero with a well-designed radius bench. For the case where all error motions are zero on average and the gauge axis is collinear with the probe (with some uncertainty), the equation reduces to

$$\langle R^2 \rangle = \langle d_{ce}^2 \rangle + \langle \delta x^2 \rangle + \langle \delta y^2 \rangle + \langle (dx^{cf})^2 \rangle + \langle (dy^{cf})^2 \rangle + \langle (dz^{cf})^2 \rangle + \text{higher order terms} \quad (4)$$

Note that the expectation values of the squared terms are not zero – these expectations are equal to the variance of the parameter<sup>8</sup>. The higher order terms may be significant in some cases. The presence of the variances in addition



to the  $d_{ce}^2$  term reflects the biases that are intrinsically present because all misalignments and errors always lead to a value of  $d_{ce}$  that is too small.

An equation for the measurand makes it possible to use a Taylor series expansion to directly calculate an estimate of the combined standard uncertainty. Rather than calculating the series expansion of the equation for  $R$ , it is easier to work with the series expansion of  $R^2$  first. We can estimate a combined uncertainty for  $R^2$  and then apply error propagation to estimate the uncertainty for  $R$ . Letting  $x = \langle R^2 \rangle$  then  $R = \sqrt{x}$ . Applying error propagation we then have<sup>9, 10</sup>

$$\mu_c = \mu_R = \frac{d(R)}{d(x)} \delta(x) = \frac{1}{2\sqrt{x}} \delta(x) = \frac{\delta(x)}{2R} = \frac{\mu_{R^2}}{2R} \quad (5)$$

Calculating the uncertainty for  $R^2$  can also be daunting. Even though it is a simple sum of terms, the occurrence of products of variables complicates matters, particularly if the expected value of many terms is zero. The simple first order Taylor series expansion may not be enough. Higher-order partial derivative terms may have to be considered. Fortunately, the terms requiring multiple partial derivatives will likely be small and can be ignored. When the error motions and misalignments are zero on average and only the first derivative terms are significant, the combined uncertainty for  $R^2$  becomes

$$\mu_{R^2}^2 = \left( \frac{\partial(R^2)}{\partial d_{ce}} \right)_{\langle d_{ce} \rangle}^2 \mu_{d_{ce}}^2 + \left( \frac{\partial(R^2)}{\partial \langle dz^{cf} \rangle} \right)_{\langle dz^{cf} \rangle}^2 \mu_{dz^{cf}}^2 \quad (6)$$

**Example:** The complexity of evaluating the expectation value of Equation 3 and estimating the uncertainty will be system dependent. We have applied the new analysis method to a radius measurement of a micro-sphere on a micro-interferometer<sup>11</sup>. For this instrument, the error motions, probe offsets, and confocal misalignments were all zero on average and uncertainties in these quantities were estimated from details of the alignment, calibration, and measurement procedures. Because error motions and misalignments are zero on average we can use Equation 4 to estimate  $R^2$  and therefore  $R$ . From consideration of our uncertainties, it is clear that the higher-order terms in Equation 4 are small and can be ignored. The uncertainty for  $R^2$  is estimated through a Taylor series expansion of Equation 3, and for our case Equation 6 can be used. The best estimate for the radius is 0.408 mm  $\pm$  0.007 mm, which represents a 1.7% combined standard uncertainty. The uncertainty is dominated by the repeatability of the measurement and the calibration of the displacement gauge.

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