A STUDY ON MODEL BUILDING OF FREE-FORM CURVES

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ABSTRACT
Based on the least-zones principle, a new mathematical modeling method established for free-form curves from the discrete sampling data, using one-parameter cubic non-uniform B-splines, is presented. Approximate errors with this method are the least. A new selection method of non-uniform fitting points, based on the equal-chordal-height method, is developed. It is proved that approximate errors of free-form curves, based on the new selection method of non-uniform fitting points, are greatly reduced. The new model building method can implement the requirement that approximate errors should be the least.

1. PREFACE
After performing fast and accurate measurement of free-form curves and surfaces with CNC-CMM in reverse engineering, a key task is to implement full and smooth reconstruction on the curves and surfaces with the acquired discrete data. At the same time, fitting accuracy of the curves and surfaces should be the best. The main method of the reconstruction on curves and surfaces is fitting method, which is divided into two methods--the interpolation method and the approximation method[1]. The approximation method, however, can perform smooth reconstruction on the curves and surfaces. Therefore, the approximate method is widely used in fitting curves and surfaces. Furthermore, the non-uniform B-Splines is one of the best tools, with which approximating errors are far less. In order to obtain a fine approximate method, we establish, based on the least-zones principle, a new mathematical model building method on curves with discrete data, using one-parameter cubic non-uniform B-splines.

2 MODEL BUILDING ON CURVES WITH LEAST APPROXIMATE ERRORS
Because of the requirements of high precision and high smoothness, it should be based on the least-zones principle in curves fitting or surfaces fitting. However, we widely use the least-squares method to substitute for the least-zones principle. Moreover, fitting performances of non-uniform B-Splines are better than those of uniform B-Splines[2][3][4]. Therefore, based on the least-squares method, we use one-parameter cubic non-uniform B-splines to establish a model building method on curves. The least-squares model building method on curves is as follows:

Given $N$ sampling points in interval $[a, b]$: $(x_i, y_i)$ $(i = 1, 2, \cdots, N)$, the usual fitting model of a curve with non-uniform B-Splines is

$$S(x) = \sum_{i=1}^{n+4} d_i B_{i,4}(x),$$

where $n$ is the number of fitting points and $B_{i,4}(x)$ is the base of the cubic non-uniform B-splines.

Then, we have the equation of residual errors at sampling points
\[ r_k = S(x_k) - y_k = d_1 B_{1,4}(x_k) + d_2 B_{2,4}(x_k) + \cdots + d_{n+4} B_{n+4,4}(x_k) - y_k \quad (k = 1, 2, \cdots, N) \]

The residual sum of squares is:
\[ F(d_1, d_2, \cdots, d_{n+4}) = \sum_{k=1}^{N} r_k^2 \]

Based on the least-squares method, the residual sum of squares should be the least i.e.
\[ \frac{\partial F}{\partial d_2} = 0, \frac{\partial F}{\partial d_2} = 0, \cdots, \frac{\partial F}{\partial d_{n+4}} = 0 \]

\[ \begin{bmatrix} \sum B_{1,4}^2 & \sum B_{1,4} B_{2,4} & \cdots & \sum B_{1,4} B_{n+4,4} \\ \sum B_{2,4} B_{1,4} & \sum B_{2,4}^2 & \cdots & \sum B_{2,4} B_{n+4,4} \\ \vdots & \vdots & \ddots & \vdots \\ \sum B_{n+4,4} B_{1,4} & \sum B_{n+4,4} B_{2,4} & \cdots & \sum B_{n+4,4}^2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n+4} \end{bmatrix} = \begin{bmatrix} \sum B_{1,4} f_k \\ \sum B_{2,4} f_k \\ \vdots \\ \sum B_{n+4,4} f_k \end{bmatrix} \]

where \( \sum B_{i,4} B_{j,4} = \sum_{k=1}^{N} B_{i,4}(x_k) \cdot B_{j,4}(x_k) \)

\( \sum B_{i,4} f_k = \sum_{k=1}^{N} B_{i,4}(x_k) \cdot f_k \)

3 EFFECTS OF DIFFERENT SELECTION METHODS OF FITTING POINTS ON APPROXIMATE ERRORS

![Fig. 1 Schematic diagrams of selection methods of fitting points](image)

a) equal-chordal-height method  
b) equal-arc-length method  
c) interpolation method

Usually, in curve fitting, fitting points are evenly selected. This selection method of fitting points is adapt to even curves but for complex curves fitting accuracy of the method is poor because equidistant fitting points can not exactly reflect the characteristics of the patterns of curves. In order to obtain a perfect selection method of fitting points, which really reflects to the characteristics of the patterns of curves, three nonuniform selection methods, which are the equal-chordal-height method, the equal-arc-length method and the interpolation method respectively, are presented. The schematic diagrams are shown in Fig. 1.

5 COMPUTER SIMULATIONS OF FITTING CURVES WITH UNIFORM AND NONUNIFORM FITTING POINTS

The units of all the numbers in following figures are the same, and the values of all the numbers are absolute values.

5.1 Relations between Fitting Errors and The Number of Uniform Sampling and Fitting Points

For the curve defined as follows, we made two groups of simulations.
\[ f(x) = \frac{100}{\sqrt{80^2 + (x - 200)^2}}, \text{ where the range of variable } x \text{ is } x \in [-400, 400] \]

We firstly keep \( n=12 \) equidistant fitting points and evenly take \( N=50 \) and \( N=80 \) sampling points in the interval respectively. Fitting error curves are shown in Fig. 4a) and 4b) respectively. Secondly, we keep \( N=80 \) equidistant sampling points and evenly take \( n=12 \) and \( n=25 \) sampling points in the interval. After
curves fitting, we’ve got series of fitting error curves shown in Fig. 5a) and 5b) respectively.

![Fig. 2 fitting errors of the curve B](image)

Fig. 2 fitting errors of the curve B

From Fig. 2, we can find the number of sampling points justly make curves more smoothly but fitting errors are not reduced significantly. From Fig. 3, we can see that the more the number of fitting points, the less fitting errors. However, the patterns of fitting error curves are not greatly changed, which fitting errors at the positions where the curvature of the curve varies greatly are great. This is to say that we can properly increase the number of fitting pints to improve the fitting accuracy of a curve.

5.2 Computer Simulations of Fitting Curves with Nonuniform Fitting Points

We use three kinds of nonuniform selection methods of fitting points, which are the equal-chordal-height method, the equal-arc-length method and the interpolation method respectively, to fit the following curve \( f(x) = \sin x \), where the range of variable \( x \) is \( x \in [0, \pi] \). To be compared, we also take the uniform selection method in fitting curves. Fitting results of the curve are shown in Fig. 8.

![Fig. 3 fitting errors of the curve A](image)

Fig. 3 fitting errors of the curve A

Compared with other methods, fitting points with the equal-chordal-height selection method can really reflect the pattern of the curve and fitting accuracy should be the best. That fitting errors shown in Fig. 4d) are the least among four kinds of methods proves this. Because fitting points based on the method are self-adaptively densely distributed at the positions where the curvature of the curve varies hard, fitting errors are significantly reduced at those positions. Thus, the fitting accuracy is greatly improved and the pattern of the fitting error curve is different from those based on other selection methods. The degree of the improvement of fitting accuracy in Fig. 4d) is relative low because the interval of curve A is relatively small.

6 EXPERIMENT OF THE APPROXIMAT MODEL BUILDING ON CURVES

In order to prove that the new model building method, we measured a standard cycloidal gear, using a trigger probe of which the diameter of the ball is 2.999\( \text{mm} \) on the CMM. The original function expression and measuring parameters of the cycloidal gear are as follows:

\[
\begin{align*}
x &= R \left( \sin \alpha - \frac{Z \cdot \sin \theta \cdot \cos \alpha}{Z \cdot \sin \theta + \cos \alpha} \right) + r_i \sqrt{K \cdot \sin \theta \cdot \cos \alpha + \sin \alpha} \\
y &= R \left( \cos \alpha - \frac{Z \cdot \cos \theta \cdot \sin \alpha}{Z \cdot \cos \theta + \sin \alpha} \right) - r_i \sqrt{K \cdot \cos \theta \cdot \sin \alpha + \cos \alpha} 
\end{align*}
\]
\[ \alpha \in \left[ -\frac{\pi}{Z_a}, \frac{\pi}{Z_a} \right], \quad K_1 = 0.5505, R_s = 109 \text{ mm}, \quad r_s = 8.5 \text{ mm}, \quad Z_a = 11, \quad Z_b = 12. \]

In fitting the curve, we took 41 sampling points and 11 fitting points. Fig. 5a-d are the fitting error curves with the equidistant selection method, the interpolation selection method, the equal-arc-length selection method and the equal-chordal-height selection method of fitting points respectively. The unit of numbers in each figure is \text{ mm} and the values of all the numbers are absolute values.

From the results of the experiment, we can find that approximate errors of the curve with the equal-chordal-height selection method of fitting points are much less than others. Approximate errors on the curve are the least. At the same time, the fitting curve keeps the same pattern of the original curve and it is smooth enough.

7 CONCLUSIONS

With careful analyses of the theories and plenty of experiments, we can conclude:

Based on the least-zones principle, it is feasible for the new modeling method, using one-parameter cubic non-uniform B-splines and least–squares fitting algorithm, to implement the reconstruction on curves of which approximate errors are the least. The influences of the distributions of uniform and non-uniform fitting points on approximate errors of the new curve models are significant. Moreover, using the equal-chordal-height selection method of non-uniform fitting points, approximate errors of the curves can be reduced greatly. With the new method, the reconstruction of curves, which approximate errors are the least, is implemented.

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